Final Research Report Submitted to the Austrian Marshall Plan Foundation

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Nicole Vorderobermeier
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1 Timeline

Research on the topic "Existence of solutions to evolution equations related to p-elastic energies" was conducted under supervision of Prof. Armin Schikorra on site at the University of Pittsburgh from January 5th until March 23rd 2020. However, the actual duration of this research project exceeds the duration of the stay, which will be discussed in the following.

In a preliminary phase, before the arrival at the University of Pittsburgh, several preparations were completed. This includes preparatory meetings with Prof. Simon Blatt, supervisor at the University of Salzburg, as well as the study of literature related to fractional versions of elastic energies, in particular so called *tangent-point energies*, regularity theory and gradient flows, i.a. [SvdM12, BR15, Man02, MM12].

The main phase of the research project took place at the University of Pittsburgh. In the beginning of the stay, discussions with the local supervisor Prof. Armin Schikorra led to the extraction of the core research question and the main hypothesis was formed. The latter roughly states higher regularity of critical points for scale-invariant tangent-point energies. This result is an intriguing novelty, as it was completely out of reach up to now, and several conclusions can be drawn from that, especially for the non-scale-invariant case. Details regarding the precise statement, its consequences, and its classification into the research landscape are treated in the upcoming sections.

In consultation and collaboration with Prof. Simon Blatt and in addition, Prof. Philipp Reiter from the Martin Luther University of Halle-Wittenberg, who is an outstanding expert on curvature energies, the mathematical argumentation was started to develop under supervision of Prof. Armin Schikorra. Amongst others, crucial methods and techniques were studied accurately, e.g. [Sch15, BRS19, Ada75, SU81, SvdM12], and calculations got carried out in detail. At least weekly discussions with Prof. Armin Schikorra provided profitable ideas and helpful advices throughout this process. Essential milestones are:

- Elaboration of suitable notions and expected statements: Section 2
- Familiarization with energy spaces: Section 3
- Proof of homeomorphisms appear as limits: Section 4
- Regularity theory (work in progress): Section 5

In the main phase of the research stay, the COVID-19 pandemic emerged and the decision to travel back to Europe approximately four weeks earlier than planned was taken, eventually. In spite of the unexpected event of the COVID-19 pandemic and its psychological strain and early return journey associated with it, the supervisors Prof. Simon Blatt from the University of Salzburg and Prof. Armin Schikorra from the University of Pittsburgh optimally ensured all necessary support to continue the collaboration in long-distance until the originally planned return and beyond that.

In the final phase, research findings are summarized in one paper in collaboration with Prof. Simon Blatt, Prof. Philipp Reiter, and Prof. Armin Schikorra, and subsequently, published. We are confident that the obtained results will lead to a publication in a high-ranking journal due to the significant research findings, not to mention its similarities to the Willmore energy, cf. the celebrated work on Willmore surfaces [Riv08] for example, and its complex methods of proof, which will be outlined later on. The completion and publishment of the paper is expected, despite the additional delay triggered by the COVID-19 pandemic, by the end of summer 2020. Finalization of the paper and follow-ups are done online via email and online meetings. The complete research period is hence expected to last for approximately one year.

2 Introduction and main results

When modelling and simulating topological effects in biology, chemistry, or physics (e.g. protein knotting) one has to make a choice. Either one explicitly models partial differential equations that incorporate different effects such as bending or self-repulsion through penalization. Or one tries to construct an energy that models the behavior at hand, and hopes that minimizing the energy, respectively following the steepest descent, delivers a realistic description. For the latter, knot energies have been introduced by Fukuhara [Fuk88] and O'Hara [O'H91, O'H92, O'H94]. One of the characteristic properties of these knot energies is that they model the topological resistance, i.e. self-repulsion, without resorting to a penalization term, but by incorporating them intrinsically, which necessarily leads to nonlocal curvature energies of fractional order.

The analytically most interesting cases of such energies are scale-invariant knot energies. They are to a certain extent an one-dimensional analogue of the Willmore energy. However, let us remark that the one-dimensionality does not simplify much the matter due to the nonlocality of the situation.

O'Hara introduced in [O'H91, O'H92, O'H94] the first class knot energies, which known as O'Hara's knot energies. They are defined as follows. Let $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be the parametrization of a closed regular Lipschitz curve in the three-dimensional Euclidean space. Technically speaking, γ is both an immersion and an embedding, therefore. The Euclidean structure of the surrounding space $\mathbb{R}^3$ induces a metric d_γ on $\gamma(\mathbb{R}/\mathbb{Z})$, namely for two points on the curve $\gamma(x), \gamma(y) \in \gamma(\mathbb{R}/\mathbb{Z})$,

$$d_\gamma(\gamma(x), \gamma(y)) := \min \left\{ \int_{[x,y]} |\gamma'|, \int_{\mathbb{R}/\mathbb{Z}\setminus[x,y]} |\gamma'| \right\}.$$

O'Hara's knot energies are then defined for any $\alpha p \geq 4$, $p \geq 2$, as

$$\mathcal{O}^{\alpha,p}(\gamma) := \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \left(\frac{1}{|\gamma(x) - \gamma(y)|^\alpha} - \frac{1}{d_\gamma(\gamma(x), \gamma(y))^\alpha} \right)^{\frac{p}{2}} |\gamma'(x)|\, |\gamma'(y)|\, dx\, dy.$$

The scale-invariant case is $\alpha p = 4$. Until very recently, among all scale-invariant O'Hara's know energies, only the so-called Möbius energy, given by $\alpha = 2$, $p = 2$, was understood at all. This is due to the celebrated work by Freedman–He–Wang, [FHW94]. They discussed existence and regularity of minimizers w.r.t. knot classes via geometric methods which crucially relied on a special property, the Möbius invariance of $\mathcal{O}^{2,2}(\gamma)$. Since their arguments are based on the Möbius invariance, and Möbius invariance is known (and most likely true) only for very special cases, there was not much progress on existence or regularity of scale-invariant knot energies for a long time. Precisely, in the two recent works [BRS16, BRS19], Blatt, Reiter, and Schikorra established the regularity theory for all scale-invariant O'Hara's knot energies $\mathcal{O}^{\alpha,p}$ for critical points and, in particular, minimizers via a completely new method. Namely, they showed that critical knots γ induce via their derivative γ' a sort of fractional harmonic map between $\mathbb{R}/\mathbb{Z}$ and $\mathbb{S}^2$. Then, with extending the tools developed for harmonic and fractional harmonic maps they completed regularity theory via arguments based on compensation effects and Harmonic Analysis.

In this work we push further the analysis of minimizers and investigate scale-invariant tangent-point energies. As in the case of O'Hara's knot energies, the scale-invariant situation is the most challenging, and up to now was completely out of reach.

Tangent-point energies were proposed first by Gonzalez and Maddocks [GM99], with variations proposed in [BGMM03, SvdM12, BR15]. They are defined for any $\gamma \in C^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ as follows

$$\begin{aligned}\mathrm{TP}^{p,q}(\gamma) &:= \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{\left|\frac{\gamma'(x)}{|\gamma'(x)|} \wedge (\gamma(x) - \gamma(y))\right|^q}{|\gamma(x) - \gamma(y)|^p} |\gamma'(x)||\gamma'(y)|\, dx\, dy \\ &= \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma'(x) \wedge (\gamma(x) - \gamma(y))|^q}{|\gamma(x) - \gamma(y)|^p} |\gamma'(x)|^{1-q}|\gamma'(y)|\, dx\, dy\end{aligned}$$

This energy is called tangent-point energy, because for $p = 2$, $q = 1$

$$R_t(x, y) := \frac{|\gamma(x) - \gamma(y)|^2}{2\,|\gamma'(x) \wedge (\gamma(x) - \gamma(y))|}$$

is the (smallest) radius of the unique circle passing through $\gamma(x)$ and $\gamma(y)$ which is tangential at $\gamma(x)$.

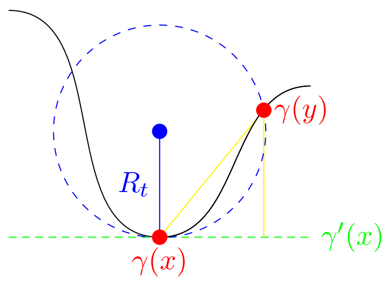

Figure 1: This picture illustrates the definition of the tangent-point function R_t.

One of the main features that motivates the tangent-point energy is that there is a natural analogue for surfaces, see [BGMM03, SvdM13]. This is also one motivation behind the present work. The analysis developed here is not only fundamental for the theory in one dimension, but may (in a future work) also be extendable to surfaces. The subcritical case of the tangent-point energies, i.e. $p > q + 2$, was discussed in [BR15]. Strzelecki and von der Mosel [SvdM12] obtained the first fundamental results in the scale-invariant case $p = q + 2$. They showed in particular that the images of curves with finite $\mathrm{TP}^{q,q+2}$-energy form a topological one-manifold. However, observe this could be a doubly-traversed line, see Example 4.1, or a non-C^1-image, see Example 4.7. These examples also show that there is an issue with even defining the notion of minimizing curves of the tangent-point energies. While the energy of the doubly-traversed line is zero, there is no regularity or the like.
Thus we restrict our interest on curves which appear as the limit of diffeomorphisms. Let us introduce the localized energy for $A \subset \mathbb{R}/\mathbb{Z}$ by

$$\mathrm{TP}^{p,q}(\gamma; A) := \int_A \int_A \frac{|\gamma'(x) \wedge (\gamma(x) - \gamma(y))|^q}{|\gamma(x) - \gamma(y)|^p}\, dx\, dy$$

In the remainder we always assume $p \geq q + 2$.
Following the spirit of an analogue idea for Willmore surfaces, [Riv08, Definition I.1], we define

Definition 2.1 (Homeomorphisms with locally small tangent-point energy)**.** Let $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a Lipschitz map and $x \in \mathbb{R}/\mathbb{Z}$.
γ is called a homeomorphism with locally ε-small tangent-point energy at x if there exists an open interval $B(x,\rho) \subset \mathbb{R}/\mathbb{Z}$ and a sequence of C^1-homeomorphism $\gamma_k : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$, $|\gamma_k'| \equiv 1$, such that

1. γ_k converges uniformly to γ in $\mathbb{R}/\mathbb{Z}$,
2. $\sup_k \mathrm{TP}^{p,q}(\gamma_k; \mathbb{R}/\mathbb{Z}) < \infty$, and
3. $\sup_k \mathrm{TP}^{p,q}(\gamma_k; B(x,\rho)) < \varepsilon$.

Let us remark that there exists homeomorphism with locally ε-small tangent-point energy γ which do not belong to C^1, see Example 4.7.
The justification for Definition 2.1 is two-fold. On the one hand, as mentioned above, there is no reason that Lipschitz maps with finite tangent-point energy are injective, indeed the tangent-point energy might be zero and still the map γ is a doubly covered interval, cf. Example 4.1. So it makes sense to consider limits of diffeomorphisms. On the other hand, sequences of curves with uniformly bounded tangent-point energy converge to homeomorphisms with locally ε-small tangent-point energy outside of at most finitely many points. In fact, this is the content of our first main result.

Theorem 2.2. *Let $p \geq q+2$, $\Lambda > 0$, $q > 1$; let $\varepsilon > 0$. Then there exists a number $K = K(q,\Lambda,\varepsilon,\delta)$ such that the following holds.*
Let $\gamma_k \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with $|\gamma_k'| \equiv 1$ and

$$\sup_{k\in\mathbb{N}} \mathrm{TP}^{p,q}(\gamma_k) < \Lambda.$$

Then there exist translations $p_k \in \mathbb{R}^3$ and a subsequence $k_i \xrightarrow{i\to\infty} \infty$ such that the curves $\tilde{\gamma}_{k_i} := \gamma_{k_i} - p_{k_i}$ uniformly converges to a Lipschitz map $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$. Furthermore, the map γ has the following properties:

- *γ is a bi-Lipschitz homeomorphism.*
- *$|\gamma'| = 1$ for a.e. $x \in \mathbb{R}/\mathbb{Z}$.*
- *$\mathrm{TP}^{p,q}(\gamma) \leq \liminf_{k\to\infty} \mathrm{TP}^{p,q}(\gamma_k)$.*
- *There is a discrete set Σ, with $\#\Sigma \leq K$ and the following properties: For any point $x_0 \in \mathbb{R}/\mathbb{Z}\backslash\Sigma$ there exists $\rho_{x_0} > 0$ such that*
  - *$\sup_k \mathrm{TP}^{p,q}(\gamma_k, B(x_0,\rho_{x_0})) < \varepsilon$ and*
  - *a subsequence $(\gamma_{k_i})_{i\in\mathbb{N}}$ weakly converges to γ in the Sobolev space $W^{1+\frac{p-q-1}{q},q}(B(x_0,\rho_x),\mathbb{R}^3)$.*

Remark 2.3. While we make no attempt to prove it explicitely, it is likely that there are examples for $p = q+2$ where the singular set Σ in Theorem 2.2 is nonempty. From the comparison with harmonic maps or the Möbius energy, an example could look as follows. Take a smooth curve of nontrivial topology. The scaling invariance (for harmonic maps the conformal invariance, for the Möbius energy the Möbius transform invariance) should allow a transform which performs a pull-tight effect where the "topology is contained in a small set", cf. [FHW94] for this notion, without changing the energy level, especially without blowing up the energy. This leads to a sequence of curves γ_k with uniformly bounded energy but with a pull-tight in the limit, where energy gets lost and, in particular, no Sobolev-convergence is doable.

Remark 2.4. Whether the curve γ above is globally Sobolev or not, is unclear to us. From 2-dimensional analogues [Hub57, Mv95] one might believe that this does not need to be true. However, observe that this is ruled out for O'Hara's knot energies.

Theorem 2.2, which we discuss in more detail in Section 4, extends the earlier results on the energy space of tangent-point energies $\mathrm{TP}^{p,q}$ with $p > q + 2$ by Blatt and Reiter [BR15] to the scale-invariant case $p = q + 2$. In this critical case $p = q + 2$, Theorem 2.2 can be interpreted as a one-dimensional analogue of a fundamental theorem of Müller and Sverak, [Mv95], who showed that surfaces with small second fundamental form w.r.t. L^2-norm can be conformally parametrized, see also earlier works by Toro [Tor94, Tor95]. We also refer to [KL12, LLT13, KS12, Riv15]. More precisely Theorem 2.2 is an analogue of Hélein's version of the Müller-Sverak theorem, [H0́2, Theorem 5.1.1], that limits of conformally parametrized 2D-maps with second fundamental form small in L^2 are either point maps or bi-Lipschitz. Said theorem was generalized for surfaces to a fractional case in [Sch18b]. What corresponds to a conformal map in 2D, i.e. $\partial\Phi \in CO(2,N)$, becomes in 1D arclength-parametrization, i.e. $|\gamma'| \equiv const$. Whereas the noncompactness of the conformal group $CO(2,N)$ is a major difficulty for the two-dimensional result, $|\gamma'| \equiv const$ in our case can be essentially reduced to the case $\gamma' \in \mathbb{S}^{N-1}$, and $\mathbb{S}^{N-1}$ is compact.
As a particular consequence of Theorem 2.2, weak local immersions with bounded tangent-point energies appear as limits of smooth minimizing sequences. Since these minimizing sequences are minimizing within topological classes, they are not necessarily global minimizing sequences of the tangent-point energy. This is why we, in spirit similar to Willmore surfaces [Riv08, Definition I.2], introduce local critical points. Observe that as of now, we have no way of hoping that the limit of minimizing sequences with finite energy is a minimizer in the same isotopy class, there may have been concentration/bubbling phenomena.

Definition 2.5 (Weak critical point)**.** Let γ be a homeomorphism with small tangent-point energy around $B(x,\rho)$ as in Definition 2.1. We say that γ is a critical point in $B(x,\rho)$ of $\mathrm{TP}^{p,q}$ if the following holds:
Let $\varphi \in C_c^\infty(B(x,\rho))$. If we set $\gamma_t := \gamma + t\varphi$, then

$$\frac{d}{dt}\Big|_{t=0} \int_{B(x,\rho)} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_t'(x) \wedge (\gamma_t(x) - \gamma_t(y))|^q}{|\gamma_t(x) - \gamma_t(y)|^p} |\gamma_t'(x)|^{1-q} |\gamma_t'(y)| \, dy \, dx = 0.$$

The notion of weak critical point as in Definition 2.5 can be justified by the following theorem: minimizing sequences converge away from finitely many points to a weak critical point.

Theorem 2.6. *Let $[\gamma_0]$ be an ambient isotopy class and let $\gamma_k \subset [\gamma_0]$ be a minimizing sequence for*

$$\Lambda := \inf_{\gamma \in [\gamma_0]} \mathrm{TP}^{p,q}(\gamma)$$

in the sense that $\gamma_k \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ are homeopmorphisms with $|\gamma_k'| \equiv 1$ and $\gamma_k(\mathbb{R}/\mathbb{Z})$ belongs to the knot class $[\gamma_0]$.
Then, up to taking a subsequence, γ_k converges uniformly to a limit map $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ which is a weak critical point in the sense of Definition 2.5 outside of finitely many points $\Sigma \subset \mathbb{R}/\mathbb{Z}$. Moreover $\#\Sigma \leq C(\Lambda)$.

Let us remark that there is no reason that minimizing sequences in isotopy classes converge globally in a strong norm, due to effects described in Remark 2.3
Now we come to our last main result, which is the regularity of critical points and thus minimizers.

Theorem 2.7. *Let $q \geq 2$. Let γ be a critical weak local immersion with small tangent-point energy around $B(x, \rho)$. Then $\gamma \in C^{1,\alpha}$ for some uniform constant $\alpha > 0$.*

Theorem 2.7 is in the spirit of [Riv08, Theorem I.3]. Note that the proof is still work in progress, however, only the last part of it remains to be elaborated in detail.
As a consequence of Theorem 2.6 and Theorem 2.7, we obtain

Corollary 2.8. *Let $[\gamma_0]$ be an ambient isotopy class and let $\gamma_k \subset [\gamma_0]$ be a minimizing sequence for*

$$\Lambda := \inf_{\gamma \in [\gamma_0]} \mathrm{TP}^{p,q}(\gamma)$$

in the sense that $\gamma_k \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ are homeopmorphisms with $|\gamma_k'| \equiv 1$ and $\gamma_k(\mathbb{R}/\mathbb{Z})$ belongs to the knot class $[\gamma_0]$.
Then, up to taking a subsequence, γ_k converges to a limit map $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ which is a weak critical point in the sense of Definition 2.5 outside of finitely many points (whose number is bounded in terms of Λ). In particular, in view of Theorem 2.7, the limit is $C^{1,\alpha}$ outside of finitely many points.
In fact, any minimizer of $\mathrm{TP}^{q,q+2}$ in any isotopy class (if it exists) can be parametrized by a $C^{1,\alpha}$-curve.

Observe that we do not treat existence in this work, which we will leave as a future project, cf. [MS20].

Outline

The outline of the remaining project report is as follows. In Section 3 we introduce the Sobolev spaces we work with and state some facts that were established in the beginning of the project work and turned out to be useful throughout the paper.

In Section 4 we outline the proof of our first main theorem Theorem 2.2 that sequences of diffeomorphisms with uniformly bounded tangent-point energy converge outside of a finite singular set. The argument is based on a gap-estimate, vaguely reminiscient of the Müller-Sverak [Mv95] or Hèlein [H0́2] result for bounded second fundamental form surfaces. A further crucial ingredient is an adaptation of the “straightness” analysis developed by Strzelecki and von der Mosel [SvdM12], which in their case leads to the fact that finite energy curves are topological 1-manifolds.

In Section 5 we discuss the regularity theory, Theorem 2.7. We follow the spirit of [BRS19] of building a bridge to harmonic map theory. Particularly, we introduce an energy $\mathcal{E}^{p,q}$ such that the arc-length parametrization of a critical knot γ of $\mathrm{TP}^{q+2,q}$ induces via its derivative γ' a critical map of $\mathcal{E}^q$ in the class of maps between $\mathbb{R}/\mathbb{Z}$ and the sphere $\mathbb{S}^2$. The energy $\mathcal{E}^q$ is structurally similar to the $W^{\frac{1}{q},q}$-Dirichlet energy, whose critical points are called $W^{\frac{1}{q},q}$-harmonic maps. For $q = 2$, techniques for regularity theory of $W^{\frac{1}{2},2}$-harmonic maps between manifolds were introduced in the pioneering work by Da Lio and Riviére, [DLR11b, DLR11a]; this was extended to $W^{\frac{1}{q},q}$-harmonic maps into spheres in [Sch15]. Here, we extend the techniques of [Sch15] to obtain the regularity for derivatives γ' of the arclength-parametrization of critical knots γ. Note that the last part presented in this section is still work in progress, but expected to be written down entirely in detail within the next weeks.

In Section 6 we summarize the outcome of the research project and observe that the obtained results serve a starting point for thrilling future projects, e.g. the hardly investigated existence of solutions to degenerate geometric evolution equations.

Finally, we point out that the proof of Theorem 2.6 is based on Theorem 2.2 combined with a fractional Luckhaus lemma and the theory for isotopy classes for Sobolev maps. However, we waive a detailed discussion here since it was not the main focus of the research stay, and refer to the preprint. Also, we remark that the theory we used and developed in this project is very delicate and technical in its details, which is why we decided to focus in this report on the main ideas and concepts of the elaborated paper. By this, we hope to make the overall direction of proof clear. Further details can be looked up in the preprint. Explanation of notation is included whenever necessary.

3 Preliminaries on Sobolev maps

In this section we recall some basic notation and properties of Sobolev maps and mention some facts that are particularly useful in proving the main statements of this project.
Let $s \in (0,1)$, $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}$ open. The fractional Sobolev space $W^{s,p}(\Omega)$ is defined as all maps $f \in L^p(\Omega)$ such that the Gagliardo semi-norm

$$[f]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{1+sp}}\right)^{\frac{1}{p}} dx\, dy < \infty.$$

For $s \in (1,2)$ the Sobolev space $W^{s,p}(\Omega)$ is defined to be the space of $f \in L^p(\Omega)$, for which $f' \in L^p(\Omega)$ and

$$[f']_{W^{s-1,p}(\Omega)} < \infty.$$

One important observation, cf. [Bla12, Lemma 2.1], is that small $W^{1+s,\frac{1}{s}}$-Sobolev norm implies a bi-Lipschitz estimate if $|\gamma'| > 0$. Namely, we have,

Lemma 3.1. *Let $s \in (0,1)$. For any $\lambda_1 > \lambda_2 > 0$ there exist $\varepsilon = \varepsilon(\lambda_1, \lambda_2, s) > 0$ such that the following holds. For any $-\infty < a < b < \infty$ and for any $\gamma \in \mathrm{Lip}([a,b], \mathbb{R}^3)$ such that*

$$\inf_{[a,b]} |\gamma'| \geq \lambda_1$$

and

$$[\gamma']_{W^{s,\frac{1}{s}}((a,b))} < \varepsilon,$$

we have

$$|\gamma(x) - \gamma(y)| \geq \lambda_2 |x-y|.$$

Let us also remark the following consequence of Lemma 3.1, which states that closed curves have minimal $W^{1+s,\frac{1}{s}}$-energy.

Corollary 3.2. *Let $s \in (0,1)$, $-\infty < a < b < \infty$. For any $\lambda > 0$ there exists $\varepsilon = \varepsilon(\lambda, a, b, s) > 0$ so that the following holds.*
Whenever $\gamma \in \mathrm{Lip}((a,b), \mathbb{R}^3) \cap C^0([a,b])$ with $\gamma(a) = \gamma(b)$ and $\inf |\gamma'| \geq \lambda$. Then $[\gamma']_{W^{s,\frac{1}{s}}([a,b])} \geq \varepsilon$.

The remaining lemmata in this section are assumed to be well-known to experts and we do not claim any originality here. Nevertheless, they are particularly useful in the context of this work.
We begin with two identifications for the fractional Sobolev space, in particular we show two equivalences to the standard $W^{s,p}$-seminorm.

Lemma 3.3 (Identification 1)**.** *Let $s \in (0,1)$, $p \in (1,\infty)$. Then for any ball $B \subset \mathbb{R}$ or $B = \mathbb{R}$ and any $f \in C_c^\infty(\mathbb{R})$,*

$$[f']^p_{W^{s,p}(B)} := \int_B \int_B \frac{|f'(x)-f'(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \approx \int_B \int_B \frac{\left|\frac{f(y)-f(x)-f'(x)(y-x)}{|x-y|}\right|^p}{|x-y|^{n+sp}}\, dx\, dy$$

The constant depends on s and p, but not on the set B or the function f.

Lemma 3.4 (Identification 2). *Let $s \in (0,1)$, $p \in (1,\infty)$. For any $g \in C_c^\infty(\mathbb{R})$ and any $B \subset \mathbb{R}$ a ball or $B = \mathbb{R}$ we have*

$$[g]^p_{W^{s,p}(B)} \approx \int_B \int_B \frac{⨍_{(x,y)} |g(x)-g(z)|^p\, dz}{|x-y|^{1+sp}}\, dx\, dy.$$

The constant depends on s and p but not on the set B or the function g.

Furthermore, we make use of the following

Lemma 3.5 (Sobolev embedding). *Let $B \subset \mathbb{R}$ a ball. For $s,t \in (0,1)$, $t < s$, and $p,q \in (1,\infty)$ with*

$$s - \frac{1}{p} \geq t - \frac{1}{q},$$

we have

$$[f]_{W^{t,q}(B)} \leq C(s,t,p,q)\ \mathrm{diam}\,(B)^{s-t-\frac{1}{p}+\frac{1}{q}}\ [f]_{W^{s,p}(B)}. \tag{3.1}$$

If $B = \mathbb{R}$ and $s - \frac{1}{p} = t - \frac{1}{q}$, then

$$[f]_{W^{t,q}(B)} \leq C(s,t,p,q)\ [f]_{W^{s,p}(B)}. \tag{3.2}$$

The constant $C(s,t,p,q)$ does not depend on f and B.

4 Homeomorphisms appear as limits: Proof of Theorem 2.2

It is easy to construct Lipschitz parametrization of curves $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ with vanishing tangentpoint energy $\mathrm{TP}^{p,q}(\gamma) = 0$, $p \geq q+2$, but with no reasonable regularity, namely $\gamma \notin C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$ and $\gamma \notin W^{1+\frac{p-1}{q},q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$.

Example 4.1. For any Lipschitz map $\tilde\gamma : \mathbb{R}/\mathbb{Z} \to [0,1/2]$ with $|\tilde\gamma'| \equiv 1$, if we set $\gamma(x) := (\tilde\gamma(x),0,0) \in \mathbb{R}^3$ then

$$|\gamma'(x) \wedge (\gamma(x)-\gamma(y))| = 0.$$

In particular, if for any $x \in \mathbb{R}/\mathbb{Z}$ there are only finitely many $y \in \mathbb{R}/\mathbb{Z}$ such that $\tilde\gamma(x) = \tilde\gamma(y)$, we have that

$$\frac{|\gamma'(x) \wedge (\gamma(x)-\gamma(y))|^q}{|\gamma(x)-\gamma(y)|^p} = 0 \quad \mathcal{L}^2\text{-a.e. } (x,y) \in (\mathbb{R}/\mathbb{Z})^2,$$

and thus $\mathrm{TP}^{p,q}(\gamma) = 0$.
For example, take $\tilde\gamma$ to be

$$\tilde\gamma(t) := \begin{cases} t & t < \frac{1}{2} \\ \frac{1}{2} - t & t \in [\frac{1}{2},1] \end{cases}.$$

Then γ' has a jump discontinuity at $t = \frac{1}{2}$ and $t = 0$. Thus $\gamma' \notin C(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$ and $\gamma' \notin W^{\frac{p-1}{q},q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$ whenever $\gamma \notin W^{1+\frac{p-1}{q},q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$ for any $p \geq q+2$ and $q \in (1,\infty)$.
It is easy to extend this example into a map γ with countably many points of non-differentiability but still $\mathrm{TP}^{p,q}(\gamma) = 0$.

See also example of k-covered circle [SvdM12, after Theorem 1.1].
Example 4.1 shows that there is no hope to classify a reasonable energy space of Lipschitz maps $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ with finite tangent-point energy. Rather we investigate the space of diffeomorphisms with finite tangent-point energy, which turns out to be more manageable – this is the content of the following Theorem 4.2 which is the main theorem of this section. In particular, Theorem 4.2 implies Theorem 2.2.

Theorem 4.2. *For any $\Lambda > 0$ and $\varepsilon > 0$ there exists a $L = L(\varepsilon, \Lambda) \in \mathbb{N}$ such that the following holds.*
Let $\gamma_k \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, $|\gamma_k'| \equiv 1$, be homeomorphisms with

$$\sup_k \|\gamma_k\|_{L^\infty} + \sup_k \mathrm{TP}^{q,q+2}(\gamma_k) \leq \Lambda$$

Then there exists a subsequence $(\gamma_{k_i})_{i\in\mathbb{N}}$ and $\gamma \in \mathrm{Lip}\,(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ such that the following holds for some finite set $\Sigma \subset \mathbb{R}/\mathbb{Z}$ with $\#\Sigma \leq L$.

1. *γ_{k_i} converges uniformly to γ and $\gamma \in \mathrm{Lip}\,(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$.*
2. *For any $x_0 \in \mathbb{R}/\mathbb{Z}\backslash\Sigma$ there exists a radius $\rho(x_0) > 0$ such that γ_{k_i} weakly converges to γ in $W^{1+\frac{p-q-1}{q},q}(B(x_0,\rho))$.*
3. *$|\gamma'| = 1$ a.e.*
4. *γ is uniformly bi-Lipschitz in $B(x_0, \rho)$ with the estimate*

$$(1 - \varepsilon)|x - y| \leq |\gamma(x) - \gamma(y)| \leq |x - y| \quad \forall x, y \in B(x_0, \rho).$$

5. *We have lower semicontinuity, namely*

$$\mathrm{TP}^{p,q+2}(\gamma) \leq \liminf_{k\to\infty} \mathrm{TP}^{p,q+2}(\gamma_k).$$

6. *γ is a bi-Lipschitz homeomorphism.*

We will prove a more detailed version of Theorem 4.2 in Proposition 4.10.
In order to prove Theorem 4.2 we proceed in several steps.

- First we prove in Section 4.1 that for the approximating sequence γ_k the local tangent-point energy is uniformly small away from a finite set Σ (we will refer to it as the "singular set") of points of energy concentration.
- In Section 4.2 we obtain the Sobolev estimate for smooth curves whenever the tangent-point energy is locally small, see Theorem 4.5, and as a consequence a bi-Lipschitz estimate. This estimate is obtained by a gap-estimate. This in particular characterizes the energy space for the tangent-point energies in the scale-invariant case.
- In Section 4.3 we adapt an argument due to Strzelecki and von der Mosel [SvdM12] to obtain a uniform estimate on global injectivity of the approximating sequence γ_k away from the singular points, see Theorem 4.9.
- In Section 4.4 we then obtain in Proposition 4.10 the convergence outside the singular set which implies Theorem 4.2.

4.1 Locally uniform smallness

In the first step we ensure that away from a discrete set we have locally uniformly small energy in the approximating sequence. The statement follows from a relatively standard covering argument, see, e.g. [SU81, Proposition 4.3 and Theorem 4.4.].

Proposition 4.3. *For any $\varepsilon > 0$, for any $\Lambda > 0$ there exists $L = L(\varepsilon, \Lambda)$ such that the following holds.*
For any sequence $\gamma_k \in \operatorname{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, $|\gamma_k'| \equiv 1$, such that

$$\sup_k \operatorname{TP}^{p,q}(\gamma_k) \leq \Lambda$$

there exists a subsequence γ_{k_i} and set $\Sigma \subset \mathbb{R}/\mathbb{Z}$ consisting of at most L points such that for any $x \in \mathbb{R}/\mathbb{Z} \backslash \Sigma$ there exists a radius $\rho = \rho_x > 0$ and an index $K \in \mathbb{N}$ such that

$$\sup_{i \geq K} \int_{B(x,\rho_x)} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_{k_i}'(x) \wedge (\gamma_{k_i}(x) - \gamma_{k_i}(y))|^q}{|\gamma_{k_i}(x) - \gamma_{k_i}(y)|^p} \, dy \, dx < \varepsilon.$$

4.2 Small local energy implies local Sobolev space estimates

The main novel ingredient underlying our argument for Theorem 4.2 is a gap estimate for Sobolev spaces with respect to the tangent-point energy.
As discussed in Example 4.1, it is impossible to control the Sobolev norm of γ in terms of the tangent-point energy of γ, $\operatorname{TP}^{p,q}(\gamma)$ without assuming a priori bi-Lipschitz estimates as it was done in [BR15]. However, this is not a viable method for the scale-invariant case $p = q + 2$ because the Bi-Lipschitz constant is not uniformly controlled as a sequence γ_k converges to γ. We turn this argument around and first a priori assume that the Sobolev norm is finite, and then conclude that this is an estimate which is uniform for sequences γ_k converging to γ.
The first step is the following gap estimate[1]

Lemma 4.4. *Let $p \in [q + 2, 2q + 1)$. Let $\gamma \in \operatorname{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, $|\gamma'| \equiv 1$. Then for any ball $B \subset \mathbb{R}/\mathbb{Z}$ of diameter less than $\frac{1}{2}$,*

$$[\gamma']^q_{W^{\frac{p-q-1}{q},q}(B)} \leq C(p,q) \operatorname{TP}^{p,q}(\gamma, B) + C(p,q) \, [\gamma']^{2q}_{W^{\frac{p-q-1}{q},q}(B)}, \tag{4.1}$$

whenever the right-hand side is finite.

The gap-estimate leads to the following control of the Sobolev norm. Observe again that we need to assume *a priori* that γ already belongs to the Sobolev space in question, which rules out the irregular curves in Example 4.1.

Theorem 4.5. *Let $q_0, p_0 > 1$, $q_1 < \infty$, $p_1 < \infty$ such that $p_1 - 2q_0 < 1$. Let $\varepsilon > 0$ then there exists $\delta = \delta(q_0, p_0, q_1, p_1, \varepsilon) > 0$ and a constant $C = C(q_0, p_0, q_1, p_1) > 0$ such that the following holds for any $p \in [p_0, p_1]$ and $q \in [q_0, q_1]$.*
Let $\gamma \in \operatorname{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, $|\gamma'| \equiv 1$, and assume that for some ball $B \subset \mathbb{R}/\mathbb{Z}$, $\operatorname{diam}(B) < \frac{1}{2}$, we have

$$\operatorname{TP}^{p,q}(\gamma, B) < \delta$$

[1] Lemma 4.4 is called a gap estimate, because it implies the following: for $\varepsilon := \left(\frac{1}{2C(p,q)}\right)^{\frac{1}{q}}$ we have either $[\gamma']^q_{W^{\frac{1}{q},q}(B)} \leq 2C(p,q) \operatorname{TP}^{q+2,q}(\gamma, B)$ or $[\gamma']_{W^{\frac{1}{q},q}(B)} \geq \varepsilon$.

and

$$\text{either } \gamma \in C^1(B) \quad \text{or} \quad [\gamma']^q_{W^{\frac{p-q-1}{q},q}(B)} < \infty. \tag{4.2}$$

Then

$$[\gamma']^q_{W^{\frac{p-q-1}{q},q}(B)} \le C(q_0,p_0,q_1,p_1)\,\mathrm{TP}^{p,q}(\gamma,B). \tag{4.3}$$

and we have the bi-Lipschitz estimate

$$(1-\varepsilon)|x-y| \le |\gamma(x)-\gamma(y)| \le |x-y| \quad \forall x,y \in B. \tag{4.4}$$

Let us also remark, for the sake of completeness, that the argument in the proof of Lemma 4.4 also gives a real classification of the energy space, if one assumes a priori bi-Lipschitz estimates (cf. [BR15, Proposition 2.4]).

Lemma 4.6. *Let $p \in [q+2, 2q+1)$. Let $\gamma \in \mathrm{Lip}\,(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, $|\gamma'| \equiv 1$ and $\gamma : B \to \mathbb{R}$ be bi-Lipschitz, i.e.*

$$(1-\lambda)|x-y| \le |\gamma(x)-\gamma(y)| \le |x-y|$$

Then for any ball $B \subset \mathbb{R}/\mathbb{Z}$ of diameter less than $\frac{1}{2}$,

$$\mathrm{TP}^{p,q}(\gamma,B) \le C(p,q,\lambda)[\gamma']^q_{W^{\frac{p-q-1}{q},q}(B)} + C(p,q,\lambda)\,[\gamma']^{2q}_{W^{\frac{p-q-1}{q},q}(B)}.$$

With the help of Lemma 4.6 we obtain

Example 4.7. There exists a homeomorphism $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ which is bi-Lipschitz, whose derivative is not everywhere continuous, but has finite tangent-point energy $\mathrm{TP}^{q,q+2}(\gamma)$ for any $q > 1$. Moreover there exists a sequence of C^∞-diffeomorphisms γ_k converging uniformly to γ with uniformly bounded tangent-point energy, i.e.

$$\sup_{k\in\mathbb{N}} \mathrm{TP}^{q,q+2}(\gamma_k) < \infty.$$

Indeed, denote by $N = (0,0,1)$ the north pole of $\mathbb{S}^2$.
Let $u \in W^{\frac{1}{q},q}([-\frac{1}{4},\frac{1}{4}],\mathbb{S}^2)\backslash C^0([-\frac{1}{4},\frac{1}{4}],\mathbb{R}^3)$ such that

$$\langle u, N\rangle \ge \frac{1}{4}$$

and u is constant for $|x| \le -\frac{1}{8}$ and $|x| \ge \frac{1}{8}$.
For example for any $\eta \in C^\infty_c((-\frac{1}{8},\frac{1}{8}),[0,1])$ with $\eta \equiv 1$ in $[-\frac{1}{16},\frac{1}{16}]$ we could set

$$u(x) = \left(\frac{1}{\sqrt{2}}\sin(\eta(x)\log\log 1/|x|), \frac{1}{\sqrt{2}}\cos(\eta(x)\log\log 1/|x|), \frac{1}{\sqrt{2}}\right).$$

Now let for $x \in [-\frac{1}{4},\frac{1}{4}]$,

$$\gamma(x) = \int_{-\frac{1}{4}}^{x} u(z)\,dz$$

Then γ is Bilipschitz in $[-\frac{1}{4},\frac{1}{4}]$ because

$$|\gamma(x)-\gamma(y)| \ge \langle \gamma(x)-\gamma(y), N\rangle = \int_{[x,y]} \langle u, N\rangle \ge \frac{1}{4}|x-y|.$$

Observe that γ' is constant around $x \approx -\frac{1}{4}$ and $x \approx \frac{1}{4}$, so γ can be smoothly extended into a closed curve on $[-\frac{1}{2}, \frac{1}{2}]$ which is a smooth 1-D manifold outside of $[-\frac{1}{4}, \frac{1}{4}]$. By Lemma 4.6 the curve γ has finite tangent-point energy $\mathrm{TP}^{q,q+2}$ but γ is not C^1 since γ' is discontinuous.

On the other hand, any regular homeomorphism $\gamma \in W^{1+\frac{1}{q},q}$ can be approximated by smooth homeomorphisms with uniformly controlled Bi-Lipschitz constant, so that in view of Lemma 4.6 the tangent-point energy $\mathrm{TP}^{q,q+2}$ is uniformly bounded.

4.3 The Strzelecki–von der Mosel argument: locally small energy implies global injectivity

In this section we provide a reformulation of a powerful argument due to Strzelecki and von der Mosel, [SvdM12]. They used it to show that the image of a curve with finite tangent-point energy for $p \geq q+2$ is a topological 1-manifold embedded into $\mathbb{R}^3$. Recall that this manifold could be the twice covered straight line, Example 4.1. We rework their argument to provide us with uniform injectivity for intervals with small energy, Theorem 4.9.

The following is essentially a reformulation (with a slight refinement) of [SvdM12, Lemma 2.1].

Lemma 4.8 (Strzelecki–von der Mosel)**.** *Let $p \geq q+2$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds.*
Let $\gamma \in \mathrm{Lip}\,(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, $|\gamma'| \equiv 1$, and assume that for some $x_0 \in \mathbb{R}/\mathbb{Z}$ and $\rho > 0$ we have

$$\int_{B(x_0,\rho)} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma'(x) \wedge (\gamma(y) - \gamma(x))|^q}{|\gamma(x) - \gamma(y)|^p} dy dx < \delta. \tag{4.5}$$

Moreover assume that there is $y_0 \in \mathbb{R}/\mathbb{Z}$ with $d := |\gamma(y_0) - \gamma(x_0)| \leq \rho$.
Then

$$\gamma(\mathbb{R}/\mathbb{Z}) \cap B_{2d}(\gamma(x_0)) \subset B_{\varepsilon d}(L(\gamma(x_0), \gamma(y_0)),$$

where $L(\gamma(x_0), \gamma(y_0))$ is the straight line containing $\gamma(x_0)$ and $\gamma(y_0)$ defined by

$$L(\gamma(x_0), \gamma(y_0)) = \{(1-t)\gamma(x_0) + t\gamma(y_0), t \in \mathbb{R}\}.$$

With help of the previous lemma, we gain an uniform estimate on global injectivity of the approximating sequence γ_k away from the singular points.

Theorem 4.9. *Let $p \geq q+2$. There exists $\delta > 0$ such that the following holds.*
Let $\gamma \in \mathrm{Lip}\,(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ be a homeomorphism, $|\gamma'| \equiv 1$, and assume that for some $x_0 \in \mathbb{R}/\mathbb{Z}$ and $\rho > 0$ we have that

$$\text{either } \gamma \in C^1(B(x_0,\rho)) \text{ or } [\gamma']_{W^{\frac{p-q-1}{q},q}(B(x_0,\rho))} < \infty. \tag{4.6}$$

Also assume

$$\int_{B(x_0,\rho)} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma'(x) \wedge (\gamma(y) - \gamma(x))|^q}{|\gamma(x) - \gamma(y)|^p} dy dx < \delta. \tag{4.7}$$

If for any $z_0 \in \mathbb{R}/\mathbb{Z}$ we have

$$|\gamma(x_0) - \gamma(z_0)| < \frac{1}{10}\rho,$$

then there exists $\bar{x} \in B(x_0,\rho)$ such that $\gamma(\bar{x}) = \gamma(z_0)$. In particular, we have $z_0 \in B(x_0,\rho)$.

4.4 Convergence

Together with the previous observations, we obtain the following convergence result outside of the singular set.

Proposition 4.10. *For any $\Lambda > 0$ and $\varepsilon > 0$ there exists a $L = L(\varepsilon, \Lambda) \in \mathbb{N}$ such that the following holds.*
Let $\gamma_k : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$, $|\gamma_k'| \equiv 1$, be C^1-homeomorhpisms with

$$\sup_k \|\gamma_k\|_{L^\infty} + \sup_k \mathrm{TP}^{q,q+2}(\gamma_k) \le \Lambda$$

Then there exists a subsequence $(\gamma_{k_i})_{i\in\mathbb{N}}$ and $\gamma \in \mathrm{Lip}\,(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ such that the following holds for some finite set $\Sigma \subset \mathbb{R}/\mathbb{Z}$ with $\#\Sigma \le L$.

1. *γ_{k_i} converges uniformly to γ and $\gamma \in \mathrm{Lip}\,(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$.*
2. *For any $x_0 \in \mathbb{R}/\mathbb{Z}\backslash\Sigma$ there exists a radius $\rho(x_0) > 0$ such that γ_{k_i} weakly converges to γ in $W^{1+\frac{p-q-1}{q},q}(B(x_0, \rho))$*
3. *$|\gamma'| = 1$ a.e.*
4. *γ_{k_i} and γ are uniformly bi-Lipschitz in $B(x_0, \rho)$ with the estimate*
$$(1-\varepsilon)|x-y| \le |\gamma_{k_i}(x) - \gamma_{k_i}(y)| \le |x-y| \quad \forall x, y \in B(x_0, \rho)\,\forall i. \tag{4.8}$$
and
$$(1-\varepsilon)|x-y| \le |\gamma(x) - \gamma(y)| \le |x-y| \quad \forall x, y \in B(x_0, \rho). \tag{4.9}$$
5. *For any point $x_0 \in \mathbb{R}/\mathbb{Z}\backslash\Sigma$ and any $y \in \mathbb{R}/\mathbb{Z}$ with $|\gamma_{k_i}(x_0) - \gamma_{k_i}(y_0)| \le \frac{1}{100}\rho(x_0)$ or $|\gamma(x_0) - \gamma(y_0)| \le \frac{1}{100}\rho(x_0)$ we have $|x_0 - y_0| \le \rho(x_0)$.*
6. *In particular, whenever $\gamma(x) = \gamma(y)$ then either $x = y$ or $\{x, y\} \subset \Sigma$.*
7. *We have lower semicontinuity, namely*
$$\mathrm{TP}^{p,q+2}(\gamma) \le \liminf_{k\to\infty} \mathrm{TP}^{p,q+2}(\gamma_k). \tag{4.10}$$
8. *$\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is a homeomorphism.*
9. *γ is globally bi-Lipschitz.*

5 The regularity theory for critical points

This section is dedicated to show $C^{1,\alpha}$-regularity of weak critical points for scale-invariant tangent-point energies $\mathrm{TP}^{q+2,q}$ with $q \ge 2$, cf. Theorem 2.7. Inspired by the investigations on critical O'Hara's knot energies in [BRS19] by means of fractional harmonic maps, cf. [Sch15], we proceed as follows:

- In Section 5.1 we relate critical knots of scale-invariant tangent-point energies $\mathrm{TP}^{q+2,q}$ to fractional harmonic maps: We first define a suitable energy $\mathcal{E}^{q+2,q}$ such that the unit tangent $u := \frac{\gamma'}{|\gamma'|}$ of critical knots γ of $\mathrm{TP}^{q+2,q}$ with constant-speed parametrization are critical maps of the energy $\mathcal{E}^{q+2,q}$ in the class of maps $v : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$. By observing that the new energy $\mathcal{E}^{q+2,q}$ is locally comparable to a $W^{\frac{1}{q},q}$-seminorm, cf. Section 4, its critical maps are indeed (essentially) fractional harmonic maps into the sphere $\mathbb{S}^2$.

- In Section 5.2 we derive the Euler-Lagrange equations of the new energies $\mathcal{E}^{q+2,q}$ for $q \geq 2$ and extract the highest order term of the Lagrangian.
- In Section 5.3 we finally treat the higher regularity of weak critical points. We outline the method of proof and provide insights into its technical details.

Before continuing with the upcoming subsections, we need to introduce some notation for integration on $\mathbb{R}/\mathbb{Z}$, cf. [BRS19, Remark 2.2]:

(1) We identify by $\rho(x, y)$ the distance of two points $x, y \in \mathbb{R}/\mathbb{Z}$ on $\mathbb{R}/\mathbb{Z}$, in particular $\rho(x, y) = |x - y| \bmod \frac{1}{2}$.

(2) If x and y are not antipodal, which means $|x - y| \neq \frac{1}{2}$, we denote by $x \triangleright y$ the shortest geodesic from x to y. Hence, we define for any $\mathbb{Z}$-periodic f

$$\oint_{x \triangleright y} f := \int_x^{\tilde{y}} f(z)\, dz,$$

where $\tilde{y} \in y + \mathbb{Z}$ such that $|x - \tilde{y}| < \frac{1}{2}$.

(3) Furthermore, we write

$$\sigma(x \triangleright y) = \operatorname{sgn} \oint_{x \triangleright y} 1.$$

That means, if $x \triangleright y$ is positively oriented, we have $\sigma(x \triangleright y) = 1$, and if $x \triangleright y$ is negatively oriented, we get $\sigma(x \triangleright y) = -1$.

(4) Now given a $\mathbb{Z}$-periodic function f, we define

$$(f)_{\mathbb{R}/\mathbb{Z}} := \int_0^1 f(z)\, dz$$

and

$$\fint_{x \triangleright y} f := \frac{\sigma(x \triangleright y)}{\rho(x, y)} \oint_{x \triangleright y} f.$$

5.1 A new energy $\mathcal{E}^{p,q}$

Our first objective in this subsection is to construct a new energy $\mathcal{E}^{p,q}$, which coincides with the tangent-point energies $\mathrm{TP}^{p,q}$ for sufficiently regular curves γ, but only depends on the first derivative γ'. We then show that any critical knot of the tangent-point energies $\mathrm{TP}^{p,q}$ parametrized by arc-length produces a critical $\mathbb{S}^2$-valued map of the new energy $\mathcal{E}^{p,q}$.
For this purpose, we recall that the tangent-point energies are given for any $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ by

$$\mathrm{TP}^{p,q}(\gamma) := \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma'(x) \wedge (\gamma(x) - \gamma(y))|^q}{|\gamma(x) - \gamma(y)|^p} |\gamma'(x)|^{1-q} |\gamma'(y)|\, dx\, dy.$$

Now we transform the cross product in the numerator by Lagrange's identity and the fundamental theorem of calculus to

$$
\begin{aligned}
&\left|\gamma'(x) \wedge (\gamma(y) - \gamma(x))\right|^2 \\
&= \left|\gamma'(x) \wedge (\gamma(x) - \gamma(y) - \gamma'(x)(y-x))\right|^2 \\
&= |\gamma'(x)|^2 |\gamma(y) - \gamma(x) - \gamma'(x)(y-x)|^2 - (\gamma'(x) \cdot (\gamma(y) - \gamma(x) - \gamma'(x)(y-x)))^2 \\
&= |y-x|^2 \left(|\gamma'(x)|^2 | \fint_{x \rhd y} \gamma'(z)\, dz - \gamma'(x)|^2 - ||\gamma'(x)|^2 - \fint_{x \rhd y} \gamma'(x) \cdot \gamma'(w)\, dw|^2 \right) \\
&= |y-x|^2 \Big(|\gamma'(x)|^2 | \fint_{x \rhd y} \gamma'(z) - \gamma'(x)\, dz|^2 - \tfrac{1}{4} | \fint_{x \rhd y} |\gamma'(x) - \gamma'(w)|^2\, dw + |\gamma'(x)|^2 \\
&\qquad - \fint_{x \rhd y} |\gamma'(w)|^2\, dw|^2 \Big).
\end{aligned}
$$

Additionally, observe that

$$
\begin{aligned}
\frac{|\gamma(y) - \gamma(x)|^2}{|y-x|^2} &= \fint_{x \rhd y} \fint_{x \rhd y} \gamma'(s) \cdot \gamma'(t)\, ds\, dt \\
&= \fint_{x \rhd y} |\gamma'(w)|^2\, dw - \frac{1}{2} \fint_{x \rhd y} \fint_{x \rhd y} |\gamma'(s) - \gamma'(t)|^2\, ds\, dt.
\end{aligned}
$$

Therefore, we can rewrite $\mathrm{TP}^{p,q}(\gamma)$ in terms of the first derivative γ' as

$$
\begin{aligned}
&\mathrm{TP}^{p,q}(\gamma) \\
&= \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{(|\gamma'(x)|^2 | \fint_{x \rhd y} \gamma'(z) - \gamma'(x)\, dz|^2 - \frac{1}{4} | \fint_{x \rhd y} |\gamma'(x) - \gamma'(w)|^2\, dw + |\gamma'(x)|^2 - \fint_{x \rhd y} |\gamma'(w)|^2\, dw|^2)^{\frac{q}{2}}}{|y-x|^{p-q}} \\
&\qquad \cdot \left(\fint_{x \rhd y} |\gamma'(w)|^2\, dw - \frac{1}{2} \fint_{x \rhd y} \fint_{x \rhd y} |\gamma'(s) - \gamma'(t)|^2\, ds\, dt \right)^{-\frac{p}{2}} |\gamma'(x)|^{1-q} |\gamma'(y)|\, dx\, dy
\end{aligned}
$$

This motivates to define the desired new energy $\mathcal{E}^{p,q}$ for any map $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ by

$$
\begin{aligned}
&\mathcal{E}^{p,q}(u) \\
&= \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \Bigg(|u(x) - (u)_{\mathbb{R}/\mathbb{Z}}|^2 | \fint_{x \rhd y} u(z) - u(x)\, dz|^2 \\
&\qquad - \frac{1}{4} \left| \fint_{x \rhd y} |u(x) - u(z)|^2\, dz + |u(x) - (u)_{\mathbb{R}/\mathbb{Z}}|^2 - \fint_{x \rhd y} |u(z) - (u)_{\mathbb{R}/\mathbb{Z}}|^2\, dz \right|^2 \Bigg)^{\frac{q}{2}} \\
&\qquad \cdot \left(\fint_{x \rhd y} |u(z) - (u)_{\mathbb{R}/\mathbb{Z}}|^2\, dz - \frac{1}{2} \fint_{x \rhd y} \fint_{x \rhd y} |u(s) - u(t)|^2\, ds\, dt \right)^{-\frac{p}{2}} \\
&\qquad \cdot |u(x) - (u)_{\mathbb{R}/\mathbb{Z}}|^{1-q} |u(y) - (u)_{\mathbb{R}/\mathbb{Z}}| \frac{dx\, dy}{\rho(x,y)^{p-q}}
\end{aligned}
$$

and we observe that the energies $\mathrm{TP}^{p,q}$ and $\mathcal{E}^{p,q}$ coincide accordingly.

Lemma 5.1. *For any knot γ with finite tangent-point energies $\mathrm{TP}^{p,q}$ and constant speed parametrization, we have*

$$
\mathrm{TP}^{p,q}(\gamma) = \mathcal{E}^{p,q}(\gamma').
$$

It remains to show that weak critical points of the tangent-point energies $\mathrm{TP}^{p,q}$, cf. Definition 2.5, indeed induce critical points into the sphere $\mathbb{S}^2$ of $\mathcal{E}^{p,q}$. As a consequence, we can make use of the machinery of proving higher regularity for fractional harmonic maps. To show the next result, we argue as in the proof of [BRS19, Theorem 2.1].

Proposition 5.2. *Let $q > 1$, $p = q + 2$ and $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a weak local immersion with small tangent-point energy $\mathrm{TP}^{q+2,q}$ around $B(x, \rho)$. Furthermore, denote the unit tangent field of γ by $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ such that $\int_{\mathbb{R}/\mathbb{Z}} u = 0$.*
If γ is a critical point of $\mathrm{TP}^{q+2,q}$ in $B(x, \rho)$, i.e. for any $\phi \in C_c^\infty(B(x, \rho), \mathbb{R}^3)$ if we set $\gamma_\varepsilon = \gamma + \varepsilon\phi$ it holds

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{B(x,\rho)} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_\varepsilon'(x) \wedge (\gamma_\varepsilon(x) - \gamma_\varepsilon(y))|^q}{|\gamma_\varepsilon(x) - \gamma_\varepsilon(y)|^p} |\gamma_\varepsilon'(x)|^{1-q} |\gamma_\varepsilon'(y)| \, dy \, dx = 0,$$

then u is a critical point of $\mathcal{E}^{q+2,q}$ around $B(x, \rho)$ in the class of maps $v : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$, i.e. for any $\phi \in C_c^\infty(B(x, \rho), \mathbb{R}^3)$ if we set $u_\varepsilon = \frac{u+\varepsilon\phi}{|u+\varepsilon\phi|}$ it holds

$$\begin{aligned}
\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{B(x,\rho)} \int_{\mathbb{R}/\mathbb{Z}} \Bigg(&|u_\varepsilon(x) - (u_\varepsilon)_{\mathbb{R}/\mathbb{Z}}|^2 \Big| \fint_{x \triangleright y} u_\varepsilon(z) - u_\varepsilon(x) \, dz \Big|^2 \\
&- \frac{1}{4} \left| \fint_{x \triangleright y} |u_\varepsilon(x) - u_\varepsilon(z)|^2 \, dz + |u_\varepsilon(x) - (u_\varepsilon)_{\mathbb{R}/\mathbb{Z}}|^2 - \fint_{x \triangleright y} |u_\varepsilon(z) - (u_\varepsilon)_{\mathbb{R}/\mathbb{Z}}|^2 \, dz \right|^2 \Bigg)^{\frac{q}{2}} \\
&\cdot \left(\fint_{x \triangleright y} |u_\varepsilon(z) - (u_\varepsilon)_{\mathbb{R}/\mathbb{Z}}|^2 \, dz - \frac{1}{2} \fint_{x \triangleright y} \fint_{x \triangleright y} |u_\varepsilon(s) - u_\varepsilon(t)|^2 \, ds \, dt \right)^{-\frac{p}{2}} \\
&\cdot |u_\varepsilon(x) - (u_\varepsilon)_{\mathbb{R}/\mathbb{Z}}|^{1-q} |u_\varepsilon(y) - (u_\varepsilon)_{\mathbb{R}/\mathbb{Z}}| \frac{dx \, dy}{\rho(x, y)^{p-q}} = 0.
\end{aligned}$$

5.2 Euler-Lagrange equations of $\mathcal{E}^{q+2,q}$

Since we are interested in obtaining higher regularity of critical knots for scale-invariant tangent-point energies, we focus our studies to the critical range $p = q + 2$ and $q \geq 2$ from now on.
In this section, we start with deriving the Euler-Lagrange equations of $\mathcal{E}^{q+2,q}$ for $q \geq 2$ and realize that the new energies $\mathcal{E}^{q+2,q}$ have a nonlinear and nonlocal Lagrangian. Furthermore, we obtain a decomposition of the Lagrangian into a term of highest order and terms of lower order, of which we will make use hereafter. The extraction of the highest order term was one of the major advances in the project.
In the following we will deal only with $\mathcal{E}^{q+2,q}$ for $q \geq 2$, for what reason we define

$$\begin{aligned}
&\mathcal{E}^q(u) := \mathcal{E}^{q+2,q}(u) \\
&= \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \Bigg(|u(x) - (u)_{\mathbb{R}/\mathbb{Z}}|^2 \Big| \fint_{x \triangleright y} u(z) - u(x) \, dz \Big|^2 \\
&- \frac{1}{4} \left| \fint_{x \triangleright y} |u(x) - u(z)|^2 \, dz + |u(x) - (u)_{\mathbb{R}/\mathbb{Z}}|^2 - \fint_{x \triangleright y} |u(z) - (u)_{\mathbb{R}/\mathbb{Z}}|^2 \, dz \right|^2 \Bigg)^{\frac{q}{2}} \\
&\cdot \left(\fint_{x \triangleright y} |u(z) - (u)_{\mathbb{R}/\mathbb{Z}}|^2 \, dz - \frac{1}{2} \fint_{x \triangleright y} \fint_{x \triangleright y} |u(s) - u(t)|^2 \, ds \, dt \right)^{-\frac{q+2}{2}} \\
&\cdot |u(x) - (u)_{\mathbb{R}/\mathbb{Z}}|^{1-q} |u(y) - (u)_{\mathbb{R}/\mathbb{Z}}| \frac{dx \, dy}{\rho(x, y)^2}.
\end{aligned}$$

We will observe that the L^2-gradient of $\mathcal{E}^q$ can be decomposed into a term of highest order, given as

$$Q_{B(x,\rho)}(u,\varphi) :=$$
$$q\int_{B(x,\rho)}\int_{\mathbb{R}/\mathbb{Z}} |\fint_{x\triangleright y} u(z)-u(x)\,dz|^{q-2}\fint_{x\triangleright y}\fint_{x\triangleright y}(u(z_1)-u(x))\cdot(\varphi(z_2)-\varphi(x))\,dz_1\,dz_2$$
$$\cdot\left(1-\frac{1}{2}\fint_{x\triangleright y}\fint_{x\triangleright y}|u(s)-u(t)|^2\,ds\,dt\right)^{-\frac{q+2}{2}}\frac{dy\,dx}{\rho(x,y)^2},$$

and the following remainders of lower order:

$$R^1_{B(x,\rho)}(u,\varphi) := \tfrac{q}{2}\int_{B(x,\rho)}\int_{\mathbb{R}/\mathbb{Z}}\left((a(0)-\tfrac14 b(0)^2)^{\frac{q-2}{2}}-a(0)^{\frac{q-2}{2}}\right)\left(1-\tfrac12 c(0)\right)^{-\frac{q+2}{2}}a'(0)\frac{dy\,dx}{\rho(x,y)^2}$$
$$R^2_{B(x,\rho)}(u,\varphi) := -\tfrac{q}{4}\int_{B(x,\rho)}\int_{\mathbb{R}/\mathbb{Z}}(a(0)-\tfrac14 b(0)^2)^{\frac{q-2}{2}}\left(1-\tfrac12 c(0)\right)^{-\frac{q+2}{2}}b(0)\,b'(0)\frac{dy\,dx}{\rho(x,y)^2}$$
$$R^3_{B(x,\rho)}(u,\varphi) := \tfrac{q+2}{4}\int_{B(x,\rho)}\int_{\mathbb{R}/\mathbb{Z}}(a(0)-\tfrac14 b(0)^2)^{\frac{q}{2}}\left(1-\tfrac12 c(0)\right)^{-\frac{q+4}{2}}c'(0)\frac{dy\,dx}{\rho(x,y)^2}$$
$$R^4_{B(x,\rho)}(u,\varphi) := -\tfrac{q+2}{2}\int_{B(x,\rho)}\int_{\mathbb{R}/\mathbb{Z}}(a(0)-\tfrac14 b(0)^2)^{\frac{q}{2}}\left(1-\tfrac12 c(0)\right)^{-\frac{q+4}{2}}d'(0)\frac{dy\,dx}{\rho(x,y)^2}$$
$$R^5_{B(x,\rho)}(u,\varphi) := q\int_{B(x,\rho)}\int_{\mathbb{R}/\mathbb{Z}}(a(0)-\tfrac14 b(0)^2)^{\frac{q-2}{2}}\left(1-\tfrac12 c(0)\right)^{-\frac{q+2}{2}}a(0)\,e'(0)\frac{dy\,dx}{\rho(x,y)^2}$$
$$R^6_{B(x,\rho)}(u,\varphi) := \tfrac{q}{2}\int_{B(x,\rho)}\int_{\mathbb{R}/\mathbb{Z}}(a(0)-\tfrac14 b(0)^2)^{\frac{q-2}{2}}\left(1-\tfrac12 c(0)\right)^{-\frac{q+2}{2}}b(0)\,(\tfrac12 d'(0)-e'(0))\frac{dy\,dx}{\rho(x,y)^2}$$
$$R^7_{B(x,\rho)}(u,\varphi) := \int_{B(x,\rho)}\int_{\mathbb{R}/\mathbb{Z}}(a(0)-\tfrac14 b(0)^2)^{\frac{q}{2}}\left(1-\tfrac12 c(0)\right)^{-\frac{q+2}{2}}((1-q)e'(0)+f'(0))\frac{dy\,dx}{\rho(x,y)^2},$$

where

$$a(0) := \left|\fint_{x\triangleright y}u(z)-u(x)\,dz\right|^2,\quad a'(0)=2\fint_{x\triangleright y}\fint_{x\triangleright y}(u(z_1)-u(x))\cdot(\varphi(z_2)-\varphi(x))\,dz_1\,dz_2,$$
$$b(0) := \fint_{x\triangleright y}|u(z)-u(x)|^2\,dz,\quad b'(0)=2\fint_{x\triangleright y}(u(z)-u(x))\cdot(\varphi(z)-\varphi(x))\,dz,$$
$$c(0) := \fint_{x\triangleright y}\fint_{x\triangleright y}|u(s)-u(t)|^2\,ds\,dt,\quad c'(0)=2\fint_{x\triangleright y}\fint_{x\triangleright y}(u(s)-u(t))\cdot(\varphi(s)-\varphi(t))\,ds\,dt,$$

and

$$d'(0)=-2\fint_{x\triangleright y}u(z)\cdot(\varphi)_{\mathbb{R}/\mathbb{Z}}\,dz,\quad e'(0)=-u(x)\cdot(\varphi)_{\mathbb{R}/\mathbb{Z}},\quad f'(0)=-u(y)\cdot(\varphi)_{\mathbb{R}/\mathbb{Z}}.$$

Lemma 5.3. *(Euler-Lagrange equations) Let $q\geq 2$ and $u:\mathbb{R}/\mathbb{Z}\to\mathbb{S}^2$ be a critical point of $\mathcal{E}^q$ around $B(x,\rho)$ in the class of maps $v:\mathbb{R}/\mathbb{Z}\to\mathbb{S}^2$ and $\fint_{\mathbb{R}/\mathbb{Z}}u=0$. Then we have for any test function $\varphi\in W^{\frac1q,q}_{\text{loc}}(B(x,\rho),\mathbb{R})$, which is also tangential, $\varphi\in T_u\mathbb{S}^2$,*

$$\delta\mathcal{E}^q(u,\varphi)=Q_{B(x,\rho)}(u,\varphi)+\sum_{i=1}^{7}R^i_{B(x,\rho)}(u,\varphi).$$

Indeed, the main term $Q_{B(x,\rho)}$ is of highest order, as we prove in the following, using the identification Lemma 3.4.

Proposition 5.4. *Let $q \geq 2$ and $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ be a critical point of $\mathcal{E}^q$ around $B(x,\rho)$ in the class of maps $v : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ and $\int_{\mathbb{R}/\mathbb{Z}} u = 0$. Then we have*

$$|Q_{B(x,\rho)}(u,u)| \approx [u]^q_{W^{\frac{1}{q},q}(B(x,\rho))}$$

with constants only depending on q.

And the remaining terms of lower order, which is obtained by taking advantage of the identification Lemma 3.4 and [BRS19, Proposition 2.5], and Sobolev embedding Lemma 3.5.

Proposition 5.5. *Let $q \geq 2$, $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ such that $\int_{\mathbb{R}/\mathbb{Z}} u = 0$ and $\varphi \in W^{\frac{1}{q},q}_{\mathrm{loc}}(B(x,\rho),\mathbb{R})$. Then we have*

$$|R^1_{B(x,\rho)}(u,\varphi)| \lesssim [u]^{2q-3}_{W^{\frac{1}{q},q}(B(x,\rho))}[\varphi]_{W^{\frac{1}{q},q}(B(x,\rho))} \qquad \text{for } 2 \leq q < 4,$$
$$|R^1_{B(x,\rho)}(u,\varphi)| \lesssim [u]^{q+1}_{W^{\frac{1}{q},q}(B(x,\rho))}[\varphi]_{W^{\frac{1}{q},q}(B(x,\rho))} \qquad \text{for } q \geq 4,$$

as well as

$$|R^i_{B(x,\rho)}(u,\varphi)| \lesssim \begin{cases} [u]^{q+1}_{W^{\frac{1}{q},q}(B(x,\rho))}[\varphi]_{W^{\frac{1}{q},q}(B(x,\rho))} & \text{for } i = 2,3\ , \\ [u]^{q}_{W^{\frac{1}{q},q}(B(x,\rho))}\|u\|_{L^\infty}\|\varphi\|_{L^1(\mathbb{R}/\mathbb{Z})} & \text{for } i = 4,5,6,7. \end{cases}$$

5.3 Regularity theory

In this section we finally run the grand machinery of showing higher regularity for (essentially) fractional harmonic maps, which correspond to our critical knots of interest. We establish a proof inspired by [Sch15] and [BRS19], but face some major obstacles due to the weak definition of critical points for scale-invariant tangent-point energies, cf. Definition 2.5, on the way.
Our main goal is to show the following

Proposition 5.6 (Decay estimate)**.** *Let $q \geq 2$ and $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ a critical map of $\mathcal{E}^{q+2,q}$ in $B(x,\rho)$. Then there exist $\varepsilon, \tau \in (0,1)$ and $N_0 \in \mathbb{N}$ such that the following holds.*
If $N \geq N_0$, $r > 0$ and $[u]_{W^{\frac{1}{q},q}(B_{2^N r})} \leq \varepsilon$, then

$$[u]^q_{W^{\frac{1}{q},q}(B_r)} \leq \tau [u]^q_{W^{\frac{1}{q},q}(B_{2^N r})} + [\tilde{u}]_{W^{\frac{1}{q},q}(\mathbb{R}/\mathbb{Z})} \sum_{k=1}^{\infty} [\tilde{u}]_{W^{\frac{1}{q},q}(B_{2^{N+k}r})} + 2^N r [\tilde{u}]^q_{W^{\frac{1}{q},q}(\mathbb{R}/\mathbb{Z})}.$$

Here $\tilde{u}$ denotes an extension u from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

This statement then implies Theorem 2.7: By iterating the decay estimate on small balls, cf. [BRS16, Lemma A.8], we obtain a $\sigma > 0$ such that

$$\sup_{r>0,\, x \in \mathbb{R}/\mathbb{Z}} r^{-\sigma} [u]_{W^{\frac{1}{q},q}(B_r(x))} \lesssim C(u).$$

Hence by employing Sobolev embedding on Morrey spaces, cf. [Ada75], we achieve $u \in C^{\tilde{\sigma}}$ for some $\tilde{\sigma} < \sigma$.
In order to attain Proposition 5.6, we begin with estimating the Gagliardo semi-norm of u on the left-hand side by an operator $\Gamma_{\beta,B}u$, which is introduced in the following. First recall that the term of highest order in the Euler-Lagrange equation of $\mathcal{E}^q$ is given by

$$\begin{aligned} Q_B(u,\varphi) &= \frac{q}{2}\int_B\int_{\mathbb{R}/\mathbb{Z}} a(0)^{\frac{q-2}{2}}\left(1-\tfrac{1}{2}c(0)\right)^{-\frac{q+2}{2}} a'(0)(u,\varphi)\frac{dy\,dx}{\rho(x,y)^2} \\ &= q\int_B\int_{\mathbb{R}/\mathbb{Z}}\left|\fint_{x\triangleright y} u(z)-u(x)\,dz\right|^{q-2}\left(1-\tfrac{1}{2}\fint_{x\triangleright y}\fint_{x\triangleright y}|u(s)-u(t)|^2\,ds\,dt\right)^{-\frac{q+2}{2}} \\ &\qquad\cdot\fint_{x\triangleright y}\fint_{x\triangleright y}(u(z_1)-u(x))\cdot(\varphi(z_2)-\varphi(x))\,dz_1\,dz_2\frac{dy\,dx}{\rho(x,y)^2}. \end{aligned}$$

As in [Sch15] and [BRS19], we now define the potential for some $0<\beta<1$

$$\Gamma_{\beta,B}u(z) := Q_B(u,|z-\cdot|^{\beta-1}),$$

following the definition of the Riesz potential I^β of order β, which is defined by

$$I^\beta f(x) = \int_{\mathbb{R}} |z-x|^{\beta-1} f(z)\,dz.$$

The inverse of the Riesz potential I^β is called the fractional Laplacian of order β, that is

$$(-\Delta)^{\frac{\beta}{2}} f(x) = c\int_{\mathbb{R}} \frac{f(y)-f(x)}{|x-y|^{1+\beta}}\,dy.$$

Applied to our situation, we hence observe that

$$Q_B(u,\varphi) = c\int_{\mathbb{R}} \Gamma_{\beta,B}u(z)(-\Delta)^{\frac{\beta}{2}}\varphi(z)\,dz. \tag{5.1}$$

Our first interim result is the following. We basically estimate the Gagliardo semi-norm by the introduced operator.

Proposition 5.7. *(Left-hand side estimates) Let B_r be an interval (motivated by weak local immersion), $\rho > 0$ such that $B_{2^L\rho} \subset B_r$ and $\frac{1}{q}-\frac{1}{p} > 0$ small. If $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ satisfies some conditions, then we have for any $\varepsilon > 0$*

$$\begin{aligned} [u]^q_{W^{\frac{1}{q},q}(B_\rho)} &\lesssim [u]_{W^{\frac{1}{q},q}(B_{2^L\rho})}\left\|\chi_{B_{2^K\rho}}\Gamma_{\frac{1}{p},B_{2^L\rho}}u\right\|_{L^{\frac{p}{p-1}}} + \varepsilon[u]^q_{W^{\frac{1}{q},q}(B_{2^L\rho})} \\ &\quad + C_\varepsilon\left([u]^q_{W^{\frac{1}{q},q}(B_{2^L\rho})} - [u]^q_{W^{\frac{1}{q},q}(B_\rho)}\right) \end{aligned}$$

for any $L, K \in \mathbb{N}$ large enough.

5.3.1 Mean Value Arguments

This subsection is meant to illustrate some technicalities, which arise in the case of the scale-invariant tangent-point energies.

Lemma 5.8. *Let $\alpha \in \mathbb{R}$ and $a, b \in \mathbb{R}$ with $|a-b| \lesssim \min\{|a|,|b|\}$. Then for any $\varepsilon \in [0,1]$,*

$$||a|^\alpha - |b|^\alpha| \lesssim |a-b|^\varepsilon \min\{|a|^{\alpha-\varepsilon}, |b|^{\alpha-\varepsilon}\}$$

Cf. [MSY20, Lemma 3.3.]
In our situation we have to deal with the expression

$$\frac{1}{|x-y|} \int_{(x,y)} ||z-z_2|^{\alpha-1} - |z-x|^{\alpha-1}| \, dz_2.$$

The following Lemma tells us, that it behaves very similarly to

$$||y-z|^{\alpha-1} - |x-z|^{\alpha-1}|.$$

Lemma 5.9. *Let $x, y, z \in \mathbb{R}$ three distinct points be inside a geodesic ball $B \subset \mathbb{R}$ and $\alpha \in (0,1)$. Set*

$$F(x,y,z) := \frac{1}{|x-y|} \int_{(x,y)} ||z-z_2|^{\alpha-1} - |z-x|^{\alpha-1}| \, dz_2.$$

- *If*

$$|x-y| \lesssim \min\{|x-z|, |y-z|\} \tag{5.2}$$

then for any $\varepsilon \in [0,1]$,

$$F(x,y,z) \lesssim |x-y|^\varepsilon \min\{|x-z|^{\alpha-\varepsilon-1}, |y-z|^{\alpha-\varepsilon-1}\}. \tag{5.3}$$

- *If*

$$|x-z| \lesssim \min\{|x-y|, |y-z|\} \tag{5.4}$$

then for any $\varepsilon \in [0,1]$,

$$F(x,y,z) \lesssim |x-z|^{\alpha-1} \lesssim |x-y|^\varepsilon |x-z|^{\alpha-\varepsilon-1}. \tag{5.5}$$

- *If*

$$|y-z| \lesssim \min\{|x-y|, |x-z|\} \tag{5.6}$$

then for any $\varepsilon \in [0,1]$,

$$F(x,y,z) \lesssim |y-z|^{\alpha-1} \lesssim |x-y|^\varepsilon |y-z|^{\alpha-\varepsilon-1} \tag{5.7}$$

For upcoming statement, we need the notation of the uncentered Hardy-Littlewood maximal function, which is given by

$$\mathcal{M}f(x) = \sup_{B(x,r)\ni y} \frac{1}{|B(y,r)|} \int_{B(y,r)} |f(z)| \, dz.$$

Let us recall the following proposition first.

Proposition 5.10. *[Sch18a, Proposition 6.6.] For any $\alpha \in [0,1]$,*

$$|u(x) - u(y)| \lesssim |x-y|^\alpha \left(\mathcal{M}(-\Delta)^{\frac{\alpha}{2}} u(x) + \mathcal{M}(-\Delta)^{\frac{\alpha}{2}} u(y) \right).$$

This implies

$$⨍_{x \triangleright y} |u(z_1) - u(x)| \, dz_1 \lesssim |x-y|^\alpha \mathcal{M}\mathcal{M}(-\Delta)^{\frac{\alpha}{2}} u(x),$$

and

$$⨍_{x \triangleright y} |u(z_1) - u(y)| \, dz_1 \lesssim |x-y|^\alpha \mathcal{M}\mathcal{M}(-\Delta)^{\frac{\alpha}{2}} u(y).$$

In the following, we need an adapted version of [Sch15, Proposition 6.3], given by

Lemma 5.11. *Let*

$$G(x,y,z) := |u(y)+u(x)-2u(z)| \fint_{x \triangleright y} ||z-z_2|^{\frac{1}{p}-1} - |z-x|^{\frac{1}{p}-1}|\, dz_2$$

and

$$H(x,y,z) := \fint_{x \triangleright y} |u(z_1)-u(x)|\, dz_1 \fint_{x \triangleright y} ||z-z_2|^{\frac{1}{p}-1} - |z-x|^{\frac{1}{p}-1}|\, dz_2.$$

Then for any $\alpha < \frac{1}{p}$ and $\varepsilon \in (0, 1-\alpha)$ such that $\varepsilon < \frac{1}{p} - \frac{\alpha}{2}$, $G(x,y,z)$ and $H(x,y,z)$ are, up to a constant, bounded from above by

$$|x-y|^{\alpha+\varepsilon} \left(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{\alpha}{4}}u(x) + \mathcal{M}\mathcal{M}(-\Delta)^{\frac{\alpha}{4}}u(y) + \mathcal{M}\mathcal{M}(-\Delta)^{\frac{\alpha}{4}}u(z)\right) k_{\frac{1}{p}-\frac{\alpha}{2}-\varepsilon,\frac{1}{p}}(x,y,z),$$

where $k_{s,\gamma}$ has the form

$$\begin{aligned} k_{s,\gamma}(x,y,z) := &\min\{|x-z|^{s-1}, |y-z|^{s-1}\} \\ &+ \left(\frac{|y-z|}{|x-y|}\right)^{\gamma-s} |y-z|^{s-n} \chi_{\{|y-z| \lesssim \min\{|x-y|,|x-z|\}\}} \\ &+ \left(\frac{|x-z|}{|x-y|}\right)^{\gamma-s} |x-z|^{s-n} \chi_{\{|x-z| \lesssim \min\{|x-y|,|y-z|\}\}}. \end{aligned}$$

Furthermore, we require the next statement for a special case in the proof of the upcoming Lemma 5.13.

Lemma 5.12. *Let $G, H : \mathbb{R} \to \mathbb{R}_+$, $\alpha, \beta \in (0,1)$ such that $\beta < \alpha < \frac{1}{p}$, and*

$$\begin{aligned} I := &\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} G(x) \fint_{x \triangleright y} |u(z_1)-u(y)|^2\, dz_1 H(z) \fint_{x \triangleright y} ||z-z_2|^{\frac{1}{p}-1} - |z-x|^{\frac{1}{p}-1}|\, dz_2 \\ &\cdot |x-y|^{-2+\alpha(q-2)}\, dx\, dy\, dz. \end{aligned}$$

Then

$$\begin{aligned} I \lesssim &\int_{\mathbb{R}} I^{\alpha(q-1)+\beta+\varepsilon-1} G\, \mathcal{M}\left(\mathcal{M}(-\Delta)^{\frac{\alpha}{2}} u \mathcal{M}(-\Delta)^{\frac{\beta}{2}} u\right) I^{\frac{1}{p}-\varepsilon} H \\ &+ \int_{\mathbb{R}} G\, I^{\alpha(q-1)+\beta+\varepsilon-1} \mathcal{M}\left(\mathcal{M}(-\Delta)^{\frac{\alpha}{2}} u \mathcal{M}(-\Delta)^{\frac{\beta}{2}} u\right) I^{\frac{1}{p}-\varepsilon} H \\ &+ \int_{\mathbb{R}} I^{\alpha(q-1)+\beta+\varepsilon-1} G\; \mathcal{M}\mathcal{M}(-\Delta)^{\frac{\alpha}{2}} u\, \mathcal{M}\mathcal{M}(-\Delta)^{\frac{\beta}{2}} u\; I^{\frac{1}{p}-\varepsilon} H \\ &+ \int_{\mathbb{R}} G\; I^{\alpha(q-1)+\beta+\varepsilon-1} \left(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{\alpha}{2}} u\, \mathcal{M}\mathcal{M}(-\Delta)^{\frac{\beta}{2}} u\right)\; I^{\frac{1}{p}-\varepsilon} H \end{aligned}$$

for any admissible $\varepsilon \in (0,1)$, $\alpha(q-1)+\beta-1 < \varepsilon < \frac{1}{p}$.

5.3.2 Right-hand side estimates

To prove the decay estimate Proposition 5.6 in the next step, we need to proceed with estimating the right-hand side of Proposition 5.7 and it becomes clear how we benefit from having introduced the operator $\Gamma_{\frac{1}{p},B} u$. In particular, we project the operator $\Gamma_{\frac{1}{p},B}u$ firstly into the linear space spanned by u and secondly into linear space orthogonal to u. More precisely, we observe by $|u| = 1$ a.e. that

$$\|\chi_{B_{2^K\rho}}\Gamma_{\frac{1}{p},B_{2^L\rho}}u\|_{L^{\frac{p}{p-1}}} \lesssim \|\chi_{B_{2^K\rho}}u \cdot \Gamma_{\frac{1}{p},B_{2^L\rho}}u\|_{L^{\frac{p}{p-1}}} + \|\chi_{B_{2^K\rho}}u \wedge \Gamma_{\frac{1}{p},B_{2^L\rho}}u\|_{L^{\frac{p}{p-1}}}.$$

Here we denote $v\wedge$ for any $v \in \mathbb{R}^3$ by the $\mathbb{R}^{3\times 3}$-matrix given by

$$v\wedge = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0. \end{pmatrix}$$

We then deal with each part of the splitting separately. However, both estimates are based on effects of integration by compensation using non-linear commutators as well as information from the Euler-Lagrange equations. The proofs also depend on the mean value arguments discussed in Section 5.3.1.

Lemma 5.13. *(Right-hand side estimates I; The orthogonal part) Assume that B_r is a geodesic ball (i.e. for all $x, y \in B_r$ we have $|x-y|_{\mathbb{R}/\mathbb{Z}} = |x-y|_{\mathbb{R}}$). Let $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ solve the Euler-Lagrange equations of Lemma 5.3 in B_r, then for any $\frac{1}{p} < \frac{1}{q}$ and $L \in \mathbb{N}$ large enough and any $\rho > 0$ such that $B_{2^{2L}\rho} \subset B_r$, we have*

$$\|\chi_{B_{2^K\rho}}u \cdot \Gamma_{\frac{1}{p},B_{2^L\rho}}u\|_{L^{\frac{p}{p-1}}} \lesssim [u]^q_{W^{\frac{1}{q},q}(B_{2^{2L}\rho})} + \sum_{k=1}^{\infty} 2^{-\sigma(L+k)}[\tilde{u}]^q_{W^{\frac{1}{q},q}(B_{2^{2L+k}\rho})}.$$

Here $\tilde{u}$ denotes an extension u from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

Lemma 5.14. *(Right-hand side estimates II; The tangential part) Assume that B_r is a geodesic ball. Let $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ solve the Euler-Lagrange equations of Lemma 5.3 in B_r, then for any $\frac{1}{p} < \frac{1}{q}$ and $K \in \mathbb{N}$ large enough and any $\rho > 0$ such that $B_{2^{20K}\rho} \subset B_r$, we have*

$$\begin{aligned}\|\chi_{B_{2^K\rho}}u \wedge \Gamma_{\frac{1}{p},B_{2^{10K}\rho}}u\|_{L^{\frac{p}{p-1}}} \lesssim [u]^q_{W^{\frac{1}{q},q}(B_{2^{20K}\rho})} &+ 2^{-\sigma K}[u]^{q-1}_{W^{\frac{1}{q},q}(B_{2^{20K}\rho})} \\ &+ [\tilde{u}]_{W^{\frac{1}{q},q}(\mathbb{R})}\sum_{k=1}^{\infty} 2^{-\sigma(K+k)}[\tilde{u}]^{q-1}_{W^{\frac{1}{q},q}(B_{2^{20K+k}\rho})} \\ &+ (2^{2K}\rho)[\tilde{u}]^q_{W^{\frac{1}{q},q}(\mathbb{R})}.\end{aligned}$$

Here $\tilde{u}$ denotes an extension u from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

The precise proof of the right-hand side estimates are expected to be written down completely within the next weeks. Then by combining the left-hand side estimates Proposition 5.7 with the right-hand side estimates Lemma 5.13 and Lemma 5.14, we finally arrive at the decay estimate Proposition 5.6.

6 Conclusion and Outlook

Overall we conclude that the research stay at the University of Pittsburgh sponsored by the Austrian Marshall Plan Foundation led to intensive and in-depth conducted investigations on the regularity theory for scale-invariant tangent-point energies.

In particular, we investigated critical points and minimizers for scale-invariant tangent-point energies $\mathrm{TP}^{p,q}$ of closed curves. We established the notion of critical weak embeddings with locally bounded tangent-point energy. We showed that a) minimizing sequences in ambient isotopy classes converge to critical weak embeddings in all but finitely many points and b) are able to show their regularity. Technically, the convergence theory a) is based on a gap-potential estimate for Sobolev spaces with respect to the tangent-point energy. The regularity theory b) is based on constructing a new energy $\mathcal{E}^{p,q}$ and proving that the derivative γ' of a parametrization of a critical curve γ is a critical map with respect to $\mathcal{E}^{p,q}$ acting on torus-to-sphere maps.

The elaborated outcome will be more than sufficient for publication in a well-established, high-ranking journal, considering its significant and long-desired results as well as its intriguing methods of proof, let alone its analogies to the Willmore energy, e.g. Riviére's outstanding work on Willmore surfaces [Riv08]. Moreover, due to is interdisciplinary nature, the research project is likely to arise great attention in various fields of research, for instance in geometric analysis, harmonic analysis, and partial differential equations, but also in knot theory or molecular biology (e.g. knotted proteins). Last but not least, thanks to the profound knowledge of Prof. Armin Schikorra, I greatly deepened my knowledge in the research area of harmonic analysis. All in all, the collaboration with Prof. Armin Schikorra culminated – as expected – in a grand success.

The outcome of the project stands out as a starting point for further fascinating projects. Immediately motivated from this work, it is desirable to prove the existence of minimizers with respect to knot classes for scale-invariant tangent-point energies. So far there exists only one existence result on scale-invariant knot energies, namely for the Möbius energy case, cf. [FHW94]. However, this existence proof is heavily based on the invariance with respect to Möbius transformations, which is presumably not available in the case of scale-invariant tangent-point energies. Moreover, it would be also fascinating to extend the regularity theory to surfaces, as there exist analogs of tangent-point energies for higher-dimensional objects, cf. [BGMM03, SvdM13].

Eventually, it is obvious to study the almost unexplored field of gradient flows for scale-invariant as well as non-scale-invariant knot energies in the next step. In particular, a certain range of sub-critical tangent-point energies leads to degenerate evolution equations, of which even non-fractional versions have been hardly investigated yet. Based on the present research project, which covered the most challenging case of tangent-point energies, i.e. the scale-invariant case, we have successfully laid the foundation and elaborated the missing link for thriving pursuing studies on the existence and regularity of solutions thereof.

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