Austrian Marshall Plan Fellowship Report:

String theoretic realizations of automorphic forms

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Abstract: We propose an exact formula for capturing the degeneracies of single center $1/4-$BPS black holes in $\mathcal{N}=4$ string compactification. A key ingredient to this formula is the existence of $1/2-$BPS bound states that arise as negative discriminant states from certain walls of marginal stability. There are subtleties arising from bound state metamorphosis that we address here. We show that the contribution from these states is finite and construct bounds on these walls of marginal stability in the tessalation of the upper half plane. We finally present data which agrees which reproduces the exact degeneracy and satisfies these bounds.

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Summary of work done

This report summarizes the work done at the Stanford Institute for Theoretical Physics during the period of 7th Jan 2019 - 7th July 2019. The focus of the research stay has exclusively been on the interface of number theory and physics. The topics investigated were relating automorphic forms to black hole entropy in string theory compactifications with publications expected as listed in . These projects also lie in a bigger class of research questions which are also being investigated with the authors of the joint publications as listed in , ensuring a continuation of collaboration and exchange between the Technical University of Vienna, Stanford University and Penn State University. A continuation of this exchange was one of the key goals as listed in the grant proposal. In this section, I present a broad overview of the academic activities at Stanford University viz., research projects started and investigated, expected publications from this stay, conferences and workshops attended, and seminars/lectures given.

List of research projects

Siegel modular form coefficients from string theoretic constructions

One of the open problems in quantum gravity is to understand the origin and explain the value of black hole entropy [1],[2][3]. String theory provides a mechanism for a class of black holes (known as BPS black holes). For such black holes, the entropy is counted by certain number theoretic objects called *automorphic forms* and from these forms, one may extract black hole entropy contributions. For the case of $\mathcal{N} = 4$ string theory, however, there are some open problems here in the sense that different techniques for computing the black hole entropy (localization and number theoretic techniques) do not entirely match [4], and physical arguments require that the entropy calculated using the different techniques be equal. In this project, we analyze why there is not an exact match between the two calculations for the $\mathcal{N} = 4,\ d = 4$ black holes and from our analysis, we obtain a technique to

1. Obtain a match for the entropy calculated using the two different techniques
2. Generate a novel technique in which these black hole degeneracies can be computed in a much faster way than the existing number theory techniques
3. Put forward a theory which allows the computation of Siegel modular forms under the action of CHL orbifolds.

Jumping loci for black holes in $\mathcal{N} = 2$ string theory

A key problem in the studying the entropy of black holes in string theory is that there is a heavy dependence on the moduli of the theory. Most theories that have studied black hole entropy insofar have studied the theory primarily using a topologically protected quantity, such as the elliptic genus, which is immune to the change in the moduli. However, there exist other quantities which do capture the moduli dependence on the theory. In the study of supersymmetric partition functions, it was pointed out in [5],[6][7][8] that for a quantity known as the Hodge elliptic genus, it is possible for the index to be sensitive to more moduli changes in the theory. These changes manifest themselves as the "appearance" of new states and hence the term jumping. In the project with S. Kachru, B. Rayhaun and R. Nally, we studied the automorphic origin of these jumping loci which involved understanding the phenomenon of jumping from an arithmetic geometric perspective. In particular, we were/are investigating whether a well known automorphic form, known as the Duke-Immamoglu-Toth form [9], is the explicit automorphic form which encodes information of such loci in string theory.

Magical super gravity, Jordan algebra & higher decomposition laws

In this project, I am currently investigating the relation between Jordan algebras, magical supergravity and higher decomposition laws [10]. The current state of this project is to realize the construction of magical supergravity [11] as a Calai-Yau compactification and study its automorphic properties from a physical perspective. Due to the heavy technicality of this project and that it is surplus to scope of what was proposed and what was achieved within the Marshall plan project proposal, I shall only discuss the project that was directly related to the original proposal.

Expected publications

The titles of publications listed below may be subject to change.

1. **"Reconciliation of black hole degeneracies from localization and mock modular forms"** *with S. Murthy (KCL), V. Reys (U. Milano Bicocca), T. Wrase (TU Wien), A. Chowdhury (IIT Bhubaneshwar & TU Wien)*

2. **"Automorphic properties of BPS jumping in $\mathcal{N} = 2$ compactifications "** *with S. Kachru, R. Nally, B. Rayhaun (all Stanford University)*

3. **"Magical supergravity, Jordan algebras and higher composition laws"** *with M. Günaydin (Penn State University), S. Kachru (Stanford), B. Rayhaun (Stanford)*

Conferences, seminars and workshops

1. Invited lecture at University of Amsterdam (February 4th 2019).
   Seminar title: *DMZ Revisited: Negative discriminant states and CHL orbifolds*

2. String theory seminar at Stanford University (March 19th 2019).
   Seminar title: *Comments on number theory and geometry techniques for stringy black holes*

3. Public talk, Menlo Park/Palo Alto
   Title: *Shower thoughts of a mad scientist: The universe, hidden realities and the edge of a new science*

4. Invited participation: Conference on Number Theory, Geometry, Moonshine & Strings III, Simons Foundation, New York (February 27th - March 1st 2019)

5. Invited participation: Workshop on Automorphic Structures in String Theory, Simons Center for Geometry and Physics, Stony Brook University, New York (March 4th - March 8th 2019).

1 From black holes to number theory: An introduction

One of the major problems facing the theory of quantizing gravity is the study of black hole microstates. Black holes are highly degenerate objects whose thermodynamic properties are mysterious and poses the setting for one of the biggest and most popularized problems in physics, the information paradox [12]. The information paradox is the apparent loss of information of the physical state of a system when the system is thrown into a black hole. From the principle of statistical physics, one may argue that a first key step in solving the information paradox is to understand the partition function of a black hole. In other words, from a Boltzmannian perspective, how does one reconcile

$$S_B \sim \ln\ W \Leftrightarrow\ S_{BH} = \frac{A}{4G_N} + c_1\ \log A + \cdots ? \tag{1.1}$$

A successful theory of quantum gravity would therefore need to compute the degeneracies of black holes or its partition function. However, since black holes are highly degenerate objects with peculiar properties, their partition functions are very difficult to compute. String theory allows us to compute the partition function as an index for certain supersymmetric black holes. These classes of black holes are extremal black hole solutions in supergravity [1] The way this can be done is by realizing that for extremal supersymmetric black holes, the index receives contribution only from certain states in the theory known as BPS states. These states are robust with the changing of the energy scale of theory. Therefore, instead of counting these states in the limit where there is a black hole solution, we may instead compute theory in the limit where the solution is a well behaved quantum field theory. In this theory, the partition function enjoys a special kind of symmetry known as modular invariance which connects us to the theory of automorphic and modular forms. Depending on the kind of theory in which the black holes are being studied, and the kind of black holes studied these automorphic forms are different. The elegance of using the automorphic form approach is that they are highly symmetric objects and one may employ very complex yet elegant techniques to study them in one fell swoop. One such technique is that of the Rademacher expansion[13].

While we have stated so far that black holes can be studied using automorphic forms, we have not mentioned how. The key object that we wish to calculate for a black hole are its degeneracies and these degeneracies are precisely the Fourier coefficients of the automorphic forms. The Rademacher expansion is a technique that allows us to compute all the degeneracies of the black hole from knowing just a few "polar" terms, i.e.,

[1] The low energy effective theory of string theory.

those terms for the black hole whose electric charge n, magnetic charge m and angular momentum ℓ satisfy the condition that $\Delta = 4mn - \ell^2 < 0$, where Δ is the discriminant of the charge vector[2]. This a technique that works remarkably well for many string theories and has been studied extensively for $\mathcal{N} = 8$ theories in 4 dimensions [14] ,[15] ,[16]. To some extent, these theories are rather uncomplicated since they are extremely constrained by supersymmetry. However, by having lower supersymmetric solutions such as theories with 16 or 8 supercharges in 4 dimensions, the problem becomes much more complicated. While the scope of $\mathcal{N} = 2$ theories is out of the scope of this proposal, I was able to investigate certain open problems in the $\mathcal{N} = 4$ case thoroughly as I shall detail.

2 Outline of the report

In this report, I shall outline the progress made towards an exact number theoretic understanding of black hole degeneracies. It should be pointed out that although I mention partition functions here, I am referring to quantities that are piece wise constant on moduli space and therefore, indices. A summary of this report is as follows:

We first begin by reviewing some requirements on the theory of modular forms, Jacobi forms and Siegel modular forms in section 3. These are examples of mathematical objects known as automorphic forms. We go over a key concept- that one can identify if whether a partition function actually does describe an object like a black hole or not. This has to do with growth properties of the Fourier coefficients of these automorphic forms. We then explain how the number theory connects to the theory of black holes. This is via the geometric construction of black holes in string theory. The automorphic forms that will be discussed in fact count special cycles on Calabi-Yau manifolds. These Calabi-Yau manifolds are precisely those manifolds on which one wraps strings and branes to get a black hole solution in the large string coupling limit. From cohomological arguments, it turns out that the count of these special cycles on the Calabi-Yau manifolds is precisely the count of the BPS states of the string theory on the given Calabi-Yau. Therefore, this mathematical machinery can count physical states in a gravitational theory.

Once this machinery is in place, we begin explaining which automorphic form corresponds to which black hole. In the candidate theories that we are considering, the black

[2] This discriminant Δ should not be confused with the Ramanujan discriminant $\Delta(\tau)$ which is a modular form related to the inverse of the 24^{th} power of the Dedekind Eta function

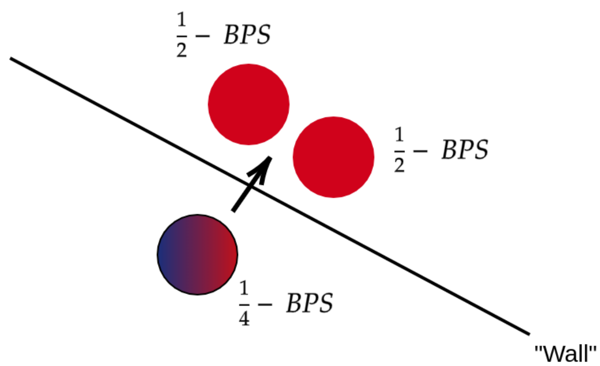

Figure 1: A graphical description of wall crossing

holes have $\mathcal{N} = 4$ supersymmetry in four dimensions. These theories have two classes of black hole solutions. To one class of these solutions viz., the $1/2-$BPS solutions, the partition function is an automorphic form on $SL_2(\mathbb{Z})$ known as the *Ramanujan discriminant*. For the other class of black hole solutions i.e., the $1/4-$BPS black holes, the partition function is the unique weight 10 automorphic form on $Sp_2(\mathbb{Z})$, known as the Igusa cusp form which is a Siegel modular form[3]. The issue comes in with the study of $1/4-$BPS black holes. The Igusa cusp form is meromorphic and this in turn influences the Fourier expansion of this automorphic form. From the above logic, it means that it computes black hole degeneracies differently, depending on the value of the meromorphic variable. This is known as *wall crossing* and it has a well defined physical interpretation when a $1/4-$BPS splits into two $1/2-$BPS states. To construct a consistent partition function, there is an elegant number theoretic prescription one may employ [17]. We review this in section 4.

However, this represents only half the story since these black hole microstates must also be computed without knowledge of the quantum field theory i.e., by the grav-

[3]We say $Sp_2(\mathbb{Z})$ here even though these are in reality 4×4 integral matrices which sometimes are also referred to as $Sp_4(\mathbb{Z})$.

itational theory itself. This should necessarily match the answer from the number theoretic side. For this, one employs a method known as localization in supergravity, reviewed briefly in section 6.

Following this, we actually note that there are still further issues with the exact matching of black hole degeneracies in supergravity and from number theory which can be attributed to the fact that there are extra monopole type solutions that influence the black hole degeneracy count. We review this problem in section 7.

The key result of this paper has been in identifying that the existing methods fail yo give the right answer at predictable values of the black hole charges. For these cases, a new theory and therefore formula has been derived. We go over this construction in section 9.

3 Pre-requisites in number theory and geometry

Since the study of black hole degeneracies involves precision counting of highly degenerate systems, it is not surprising to expect there to be a number theoretic setting. In this section, we review all the relevant number theory and geometry required for the understanding of this report.

Definition 3.1. *A modular form (MF) of weight k under $SL_2(\mathbb{Z})$ is a function on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} | \Im\tau > 0\}$ such that*

$$f(\tau) \to f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

A modular form is a periodic function in τ and therefore admits a Fourier transform: $f(\tau) = \sum_n a(n)q^n,\ q = e^{2\pi i\tau}$ and is bounded as $\Im\tau \to \infty$. A modular form ($\in M_k$) is a *cusp form* ($\in S_k$) if $a(0) = 0$ in which case the MF vanishes as infinity. A MF is weakly holomorphic ($\in M_k^!$) if $a(n) = 0\ \forall n < -N$. It is precisely these objects that will be of importance to us owing to the growth properties of the Fourier coefficients in the Fourier expansion. Weakly holomorphic MF's exhibit Hardy-Ramanujan type growth of Fourier coefficients i.e., they have an exponential growth.

Remark 3.1. *The exponential growth of these coefficients is relevant since it mimics the growth of degeneracies of a Boltzmann type entropy formula (Cardy formula for CFT's.) In other words, a polar term of the type q^{-n} is required to get a Boltzmann type system.*

Holomorphic modular forms on $SL_2(\mathbb{Z})$ are of even weight, $m \geq 4$, and are completely generated by two functions known as Eisenstein series of weight 4 and 6. These are defined as

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n},$$
$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} \tag{3.1}$$
$$\Delta(\tau) = \frac{E_4^3(\tau) - E_6^2(\tau)}{1728} = q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

where $q = e^{2\pi i \tau}$, $\Delta(\tau) = \eta(\tau)^{24}$ is the Ramanujan discriminant, and the Dedekind-η function is given by $\eta(\tau)$.

Definition 3.2. *A Jacobi form $\phi(\tau, z)$ of weight k and index m is a function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ which is modular in τ and elliptic in z*

$$\phi(\tau, z) \to \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{\frac{2\pi i m c z^z}{c\tau + d}} \phi(\tau, z) \tag{3.2}$$
$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{3.3}$$
$$\phi(\tau, z) \to \phi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2 \tau + 2\lambda z)} \phi(\tau, z) \forall \lambda, \mu \in z. \tag{3.4}$$

Since a JF is periodic in both its variables, it admits a Fourier transform: $\phi(\tau, z) = \sum_{n,\ell} c(n, \ell) q^n y^\ell$ where $c(n, \ell) = C(4mn - \ell^2, \ell) = C(\Delta, \ell)$ is fixed by $\ell \mod 2m$. A JF can be either a simple JF ($c(n, \ell) = 0$ if $\Delta < 0$), a Jacobi cusp form ($c(n, \ell) = 0$ if $\Delta \leq 0$), a weak JF ($c(n, \ell) = 0$ if $n < 0$) or a weakly holomorphic JF ($c(n, \ell) = 0$ if $n \leq n_0$). A JF in addition also has a theta decomposition given by

$$\phi(\tau, z) = \sum_{\ell \in \mathbb{Z}} q^{\ell^2/4m} h_\ell(\tau) y^\ell, \; y = e^{2\pi i \ell z} = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h_l(\tau) \vartheta_{m,\ell}(\tau, z) \tag{3.5}$$
$$h_\ell(\tau) = \sum_{\Delta} C(\Delta, \ell) q^{\Delta/4m}, \; \ell \in \mathbb{Z}/2m\mathbb{Z}, \; \vartheta_{m,\ell}(\tau, z) \sum_{r \in \mathbb{Z}, r \equiv \ell \mod 2m} q^{r^2/4m} y^r, \tag{3.6}$$

where $\vartheta_{m,\ell}(\tau, z)$ is a JF on a subgroup of $SL_2(\mathbb{Z})$ of weight 1/2 and index m. $h_\ell(\tau) = (h_1(\tau), \cdots, h_{2m}(\tau))$ is a vector valued modular form of weight $k - 1/2$. This theta decomposition of a JF into vector values modular forms is essential for the Rademacher expansion. In fact, type of MF $h_\ell(\tau)$ is depends on what kind of a JF $\phi(\tau, z)$ is.

Remark 3.2. *Once again, the polar term ($\Delta < 0$) allows for exponential growth of coefficients at infinity.*

The main object governing the theory of $1/4-$BPS black holes are Siegel modular forms. They are essentially genus 2 modular forms in which $SL_2(\mathbb{Z})$ is enhanced to $Sp(2,\mathbb{Z})$. The Siegel UHP is defined as $\mathbb{H}_2 = Mat_{2\times 2}$ of the form $M = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}$.

Definition 3.3. *A SMF $F(M)$ of weight k is a function from the Siegel UHP to the complex plane such that*

$$F\left(\frac{AM+B}{CM+D}\right) = \det(CM+D)^k F(M),$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an element of $Sp_2(\mathbb{Z})$.

A SMF admits two different expansions which are related to each other. The first is the Fourier series owing to periodicity of the SMF which is given by

$$F(M) = \sum_{m,n,\ell;\Delta\geq 0} a(m,n,\ell) q^n y^\ell p^m \tag{3.7}$$

and the second is the Fourier Jacobi series given by

$$F(M) = \sum_m c(n,\ell)\phi_m(\tau,z)p^m. \tag{3.8}$$

Remark 3.3. *For all the automorphic forms that will be discussed in this report, the key point is always that the Fourier coefficients of a $q^n y^\ell p^m$ term are precisely the number of black hole microstates at electric charge n, magnetic charge m and angular momentum ℓ.*

4 Degeneracies of dyonic black holes in $\mathcal{N} = 4$ string theory

In [18], Dijkgraaf, Verlinde and Verlinde conjectured a formula for counting the degeneracies of dyonic $1/4-$BPS states in four-dimensional $\mathcal{N} = 4$ string theory. Due to string-string duality [19], this theory can be described either in terms of heterotic string theory compactified on T^6 , or in in terms of type II string theory compactified on $K3 \times T^2$. Both theories eventually give the same theory in 4 dimensions viz., a supersymmetric theory with 16 supercharges. This theory admits two types of black hole solutions, the $1/2-$BPS solution and the $1/4-$BPS solution. Both these solutions are charged under 28 gauge fields of the T-moduli space of the theory, with further symmetry under the S-moduli space coset. These groups are the T and S–duality groups of the theory and are given by the cosets

$$\begin{aligned} S:&\ SL(2,\mathbb{Z})\setminus SL(2,\mathbb{R})\,/\,O(2,\mathbb{R}) \\ T:&\ O(6,22,\mathbb{Z})\setminus O(6,22,\mathbb{R})\,/\,O(6,\mathbb{R})\times O(22,\mathbb{R}). \end{aligned} \tag{4.1}$$

These black holes are dyonic states in the sense that they possess both electric and magnetic charges, with charges in the integral second cohomology lattice of $\Gamma_{6,22}\oplus\Gamma_{6,22}$. When the string coupling is tuned up to $g_s \gg 1$,we have a BMPV black hole [20] at the center of a Taub–NUT space with branes wrapping the various cycles of the compactification geometry with Kaluza–Klein monopole along a circular direction. Depending on whether there is momentum along one of the cycles of T^2, we end up with a $1/2-$BPS black hole (zero momentum) or a $1/4-$BPS black hole (with momentum). This is the type II frame of the theory and one can compute the leading contribution to the entropy and its asymptotic behaviour from a Cardy type formula [21]. In the heterotic description, there is a translation of the charges between the black hole description and the gauge description and one can construct invariants from the charges. There are the three T-duality invariants viz., the electric charge n, magnetic charge m and angular momentum, ℓ of the black hole. The counting problem is much more tractable here.

4.1 $1/2-$BPS black holes

For the case of $1/2-$BPS black holes, one simply counts the $1/2-$BPS states in the theory. In the heterotic description, this corresponds to counting those stringy excitations dual to the $1/2-$BPS states. These are known are *Dabholkar-Harvey* [22] states and have an automorphic form associated with them. This is simply the *Ramanujan discriminant*,

$$\frac{1}{\Delta(\tau)} = \frac{1}{\eta(\tau)^{24}} = q^{-1}\prod_{n\geq 1}\frac{1}{(1-q^n)^{24}}, \quad q := e^{2\pi i\tau}. \tag{4.2}$$

The Fourier expansion of the above form is given by

$$\frac{1}{\Delta(\tau)} = \sum_{m\geq -1} a_n q^n, \tag{4.3}$$

and the a_n, the Fourier coefficients of the Fourier expansion of the Ramanujan discriminant are precisely the black hole degeneracies of $1/2-$BPS black holes at an electric or magnetic charge of n. The reason that this is either electric or magnetic is because one can always perform a transformation on a $1/2-$BPS state which takes them into a purely electric or purely magnetic frame. There are two things of interest here:

- That the counting function for the $1/2-$BPS state, a modular form on $SL_2(\mathbb{Z})$has no complex poles and therefore experiences no wall crossing phenomena, a phenomena when it is energetically favourable for two $1/2-$BPS states to form a $1/4-$BPS bound state. This means that they exist everywhere in moduli space.
- That, as we shall see, these black holes control the degeneracy of single center $1/4-$BPS black holes which is a very surprising fact. Apriori, a $1/2-$BPS black hole should not contain much information about the degeneracy of single center $1/4-$BPS black holes except at those special loci in moduli space where there is wall crossing.

This will in fact be the key take away from this research project: that there is a technique from which one can extract the degeneracies of single center $1/4-$BPS black holes for certain charge configurations is a much easier and faster manner. Mathematically, this translates to stating that one may extract the Fourier coefficients of certain mock modular objects and Siegel modular forms in a very efficient manner from certain cycles or walls of marginal stability in the upper half plane, the set of complex numbers whose imaginary part is strictly positive.

4.2 $1/4-$BPS black holes

As stated before, these are black holes which in the type II frame carry momentum on one of the cycles of the 2-torus, thereby "puffing up" the black hole and giving it an area. A well known and well studied conjecture by [18] states that the degeneracy of such black holes is counted by the Igusa cusp form which is the unique weight 10 automorphic form on $Sp_4(\mathbb{Z})$ The proposed microscopic counting formula for $1/4-$BPS degeneracies $d_{1/4}$ is given by

$$d_{\frac{1}{4}}(m,n,\ell) = (-1)^{\ell+1} \int_{\mathcal{C}} d\tau \ d\sigma \ dz \ e^{-2\pi i(\tau n+\sigma m+z\ell)} \frac{1}{\Phi_{10}(\tau,\sigma,z)}, \tag{4.4}$$

where

$$\begin{aligned} \frac{1}{\Phi_{10}(\tau,\sigma,z)} &= e^{-2\pi i(\tau+\sigma+z)} \prod_{\substack{j,k,\ell\in\mathbb{Z},4k\ell-j^2\geq -1 \\ k,\ell\geq 0, j<0 \text{ for } k=\ell=0}} \left(1-e^{2\pi i(k\tau+\ell\sigma+jz)}\right)^{-c\left(4k\ell-j^2\right)} \\ &= \sum_{k,\ell,j} (-1)^{j+1} g(k,\ell,j) e^{2\pi i(k\tau+\ell\sigma+jz)}. \end{aligned} \tag{4.5}$$

In(4.5) above, the Igusa cusp form is expressed in terms of the Borcherds product representation which is obtained by the multiplicative lift of the elliptic genus of $K3$

and is defined as

$$Z_{ell}^{(K3)} = \sum_{i=2}^{4} \left(\frac{\vartheta(\tau, z)}{\vartheta(\tau, 0)} \right)^2 = \sum_{k,j} c(4k - j^2) q^k, y^j, \ y := e^{2\pi i z}, \tag{4.6}$$

where ϑ_i's are the Jacobi theta functions and admits a Fourier expansion owing to periodicity of the function in both (τ, z) variables. In fact the coefficients in the exponent of the product representation of (4.5) are precisely the Fourier coefficients of the elliptic genus of $K3$. It should also be stated that the Igusa cusp form admits a Saito-Kurokawa lift or an additive lift which is defined from the action of Hecke operators on the elliptic genus of $K3$. Thus, the connection to $K3$ is apparent here. This is because of the fact that the $\mathcal{N} = 4$ theory is described by a sigma model whose target space is precisely a deformation of m copies of $K3$[4] The counting function as defined above is in some an index[5] of dyons carrying the charges (m, n, ℓ). The $\mathcal{C}$ indicates that the Fourier transform must be taken along a certain contour. This is because if we want to compute only the honest $1/4-$BPS states, we need to ensure that this contour does not cross any pole in the Siegel upper half plane [23],

$$\tau_2, \sigma_2 > 0, \quad \tau_2\sigma_2 - z_2^2 > 0. \tag{4.7}$$

When one encounters a pole in the partition function Φ_{10}^{-1}, there is a wall crossing phenomena where two $1/2-$BPS states may bind to form an effective $1/4-$BPS state due to energetically favourable conditions, or vice versa[6]. The moduli space is divided into chambers separated by walls of marginal stability. In a given chamber the degeneracy for a given charge vector is constant, and it jumps as one crosses a pole in moving to another chamber.

Dyonic $1/4-$BPS in the above string theory have a macroscopic supergravity description in terms of black hole solutions. It is known that multi-centered black holes can exist depending on the region of the moduli space and the values of the charges (m, n, ℓ) [26]. Therefore, this means that unlike the $1/2-$BPS states, $1/4-$BPS states do not exist everywhere in moduli space. To count the degeneracies of honest $1/4-$BPS states, one my devise a technique in which the macroscopic supergravity and the microscopic theory yield the same answer. Until recently, such a technique was not known and it was moreover unclear if whether these degeneracies would be integral under a given technique [27].

[4]This is known either as the symmetric product of $K3$ or the Hilbert scheme of $K3$.

[5]This is because the index contribution depends only on $n_V - n_H$. In other words, even when there is wall crossing, the decay products of the dyons still maintain $n_V - n_H$.

[6]We only focus on the case where there is a two-center breakdown since any other breakdown occurs only at higher codimension walls [24, 25].

5 Microscopics of $1/4-$BPS black holes

The index of $1/4-$BPS states as defined by Φ_{10}^{-1} has automorphic symmetry. To extract the single-centered degeneracies from this automorphic form, there is an elaborate and elegant technique that was proposed in [17]. essentially, this technique involves performing a Fourier–Jacobi expansion of $\frac{1}{\Phi_{10}(\tau,\sigma,z)}$ as

$$\frac{1}{\Phi_{10}(\tau,z,\sigma)} = \sum_{m\geq -1} \psi_m(\tau,z)\, e^{2\pi i m\sigma} \,. \tag{5.1}$$

The functions $\psi_m(\tau,z)$ are Jacobi forms of index $m,\ k=-10$ that are meromorphic in z. This meromorphicity corresponds precisely to the $1/4-$BPS states partition function that undergo wall-crossing phenomenon. The single-centered degeneracies are obtained by subtracting the contribution of multi-centered states, $\psi_m^P(\tau,z)$, from $\psi_m(\tau,z)$. The *polar piece* $\psi_m^P(\tau,z)$ that contains the multi-centered contributions is proportional to an Appell-Lerch sum [17]

$$\psi_m^{\mathrm{P}}(\tau,z) = \frac{p_{24}(m+1)}{\eta^{24}(\tau)} \sum_{s\in\mathbb{Z}} \frac{q^{ms^2+s}\, y^{2ms+1}}{(1-q^s y)^2} \,, \tag{5.2}$$

where $p_{24}(m+1)$ is the coefficient of q^m in the Fourier expansion of $\Delta^{-1}(\tau)$. The Appell-Lerch sum is essentially an averaging function that averages the residues (= the multicenter contributions over walls of marginal stability contributing to the positive discriminant black holes). Since it has the polar structure of $\Phi_{10}(\tau,\sigma,z)$, $\psi_m^P(\tau,z)$ exhibits wall-crossing. Subtracting $\psi_m^P(\tau,z)$ from the functions $\psi_m(\tau,z)$ in (5.1), we obtain the generating function of degeneracies of single-centered positive discriminant $1/4-$BPS states,

$$\psi_m^{\mathrm{F}}(\tau,z) := \psi_m(\tau,z) - \psi_m^{\mathrm{P}}(\tau,z) \,. \tag{5.3}$$

As shown in [17], the function $\psi_m^{\mathrm{F}}(\tau,z)$ has important properties, the most relevant for us is that it still employs a Fourier expansion

$$\psi_m^{\mathrm{F}}(\tau,z) = \sum_{n,\ell} c_m^{\mathrm{F}}(n,\ell)\, q^n\, y^\ell \,. \tag{5.4}$$

Its Fourier coefficients capture the single-centered degeneracies for $1/4-$BPS black holes. In order to compute all the Fourier coefficient of the finite mock Jacobi form, one can use a generalization of the Hardy-Ramanujan-Rademacher formula suited for mixed mock modular forms, first put forward in [28] and later stated analytically in [29]. The general idea of this formula is that in order to compute the Fourier coefficient of positive discriminant states (i.e., the degeneracy of single center $1/4-$BPS black holes at positive discriminant), one only needs the negative discriminant or polar coefficients of $c^{\mathrm{F}}_m(n,\ell)$ i.e., $\widetilde{c}_m(n,\ell)$, $\Delta < 0$, and the theta nullwerthe $\vartheta_{m,\ell}(\tau,0)$. The coefficients hereby obtained expressed as a convergent infinite sum of I-Bessel functions, which requires only the polar terms as input. These polar terms are precisely $1/2-$BPS black hole degeneracies. The siginificance of this is that the single-centered $1/4-$BPS degeneracies are completely determined by the negative discriminant states of the mock Jacobi form ψ^{F}_m and thereby by $1/2-$BPS black holes.

6 Localization of $\mathcal{N}=4$ supergravity and black hole degeneracy

While section 4 concerns the stringy (microscopic) aspects of $1/4-$BPS states, a supergravity (macroscopic) description for the microstates of 1/4-BPS black holes has been put forward in recent years. We shall review it here all but briefly. To obtain the macroscopic version of the black hole entropy, one needs to perform localization of the $\mathcal{N}=4$ supergravity quantum entropy function [30][31][32]. Localization is a very useful tool in studying supersymmetric theories and we refer the reader to consult [33] for references on localization in supersymmetric gauge theories.

Here, the result of localization is that the entropy function of the black hole can be expressed as an infinite sum of Bessel functions and the coefficients of these Bessel functions are again the macroscopic degeneracies of the $1/4-$BPS black holes [14][4][15]. Localization of the quantum entropy function successfully reproduces the degeneracy of single center black holes with positive discriminant and most negative discriminant states, there are instances where this matching between the microscopic and macroscopic paradigms breakdown [4].

As it turns out, there is a clear reason for this mismatch owing to the existence of multicenter states that still remain in $\psi^F_m(\tau,z)$ despite the removal of wall crossing effects via subtracting the Appell-Lerch sum. This discovery will be reviewed in the remainder of the report.

7 Negative discriminant states and walls of marginal stability

As seen in the previous two sections, there are two independent ways of computing the single-centered $1/4-$BPS black hole degeneracy from a microscopic and a macroscopic formalism. These two methods are on occasion in disagreement with each other which we find can be ameliorated by a meticulous study of negative discriminant states.

It was noted in [34] that all negative discriminant $1/4-$BPS states are bound states of two $1/2-$BPS states i.e., there are remaining two center contributions in the microscopic description. Although these states are required to compute the degeneracies from the Rademacher method, their presence will ultimately result in a mismatch between the localization and microscopic paradigms due to subtleties that we discuss in the coming sections. To make progress towards a consistent $1/4-$BPS single-center degeneracy generating function, a first step is to understand which walls of marginal stability contribute towards the degeneracy of negative discriminant states. The first step we undertook towards this is to understand the construction of walls that tessellate the upper half plane. We noted that the walls of marginal stability have interesting properties, following generalization of [34]:

1. To every wall of marginal stability is an associated $SL_2(\mathbb{Z})$ matrix.
2. For every wall, given an original charge vector, there is an associated decay. From these decay products, we may read off the T–dual invariants of the decay.
3. These T-dual invariants of the decay are precisely the charges of the decay products which we call $(m_\gamma, n_\gamma, \ell_\gamma)$ for a wall γ.
4. We discover that there is a precise mathematical technique to identify which wall contributes to the black hole entropy.
5. However, due to the rather primitive or un-refined nature of the elliptic genus of $K3$, there are ambiguities in the theory when the decay products have electric and/or magnetic charges = -1. In this case, there is a phenomenon known as metamorphosis in which the black holes states "morph" into one another [35]. Because of this subtle process, we discover that there is a mismatch between the macroscopic and microscopic degeneracies of the black hole. Therefore, to get the exact black hole degeneracy, we derive a formula that consistently takes into account the effects of metamorphosis and wall crossing.

8 Summary of main results

For the case of positive discriminant $1/4-$BPS dyons, the exact degeneracy is precisely given by the Fourier coefficients of $\psi^F_m(\tau, z)$, and this is in agreement with the results from the localization of the quantum entropy function [4]. However, there is a mismatch between certain Fourier coefficients of $\psi^F_m(\tau, z)$ and the localization results for precisely those dyons with negative discriminant. This is attributed to the fact that for the case of negative discriminant dyons, $\psi^F_m(\tau, z)$ still contains bound states of $1/2-$BPS dyons. While this is surprising at first, this is straightforward to see why. $\psi_m^F(\tau, z)$ is constructed from the Jacobi form $\psi_m(\tau, z)$ by subtracting a term containing the Appell-Lerch sum which is an average over the residues of the meromorphic Jacobi form over those walls which run off to complex infinity. For the case of positive discriminant states, these are the only walls which could result in wall crossing phenomena and thus affect the value of the black hole degeneracy in the attractor contour. Removing the polar piece for positive discriminant states thus represents the end of the story. However, when the states are of negative discriminant, there are also possible contributions to the black hole degeneracy from other states and walls. The formula which we derive captures this. Furthermore, we also note that the formula we derive can be used to exactly and efficiently compute the Fourier coefficients of Siegel modular forms at computational speeds much faster than known techniques.

9 Towards a modified formula for black hole degeneracy

Given the construction of walls of marginal stability in and the requirement for consideration of negative discriminant states, we can derive a modification to the index formula given in [4, 34]. Since the actual derivation is elaborate, we refer the reader to the following publication for the details of the formula. To account for the correct black hole entropy, we need to consider two important effects:

1. That there are still bound states of $1/2-$BPS black holes if the charge vector has $\Delta = 4mn - \ell^2 < 0$ coming from those walls which have finite length.
2. These bound states for special values of charges can "morph" into one another (known as bound state metamorphosis). Therefore, there will be many walls that yield the same index contribution to a chamber which need to be identified with each other.

The expectation from the modified formula is an agreement with the degeneracies obtained from localization. Therefore, from the Fourier coefficients of the mock modular forms $\psi_m^F(\tau, z)$, one would need to further remove a contributions from certain negative discriminant states.

9.1 A proposal for a modified formula for black hole degeneracy

The technique we use is by starting to notice that the issue of whether or not the charge breakdown at any given wall contributes to the total $1/4-$BPS single center black hole degeneracy depends on the values of the transformed charges $(n_\gamma \geq -1, m_\gamma \geq -1)$ and the value of the angular momentum in particular. This is because the angular

momentum of the black hole either spins it into the right direction of where we want to compute the black hole degeneracy, or away from it. By solving for the requirement that we maintain that the discriminant of the charges, regardless of the breakdown is negative and a U-duality invariant, we analytically solve for the possible infinite set of contributions to the black hole entropy. We are also able to numerically and (in almost all cases, analytically) show that given (m, n, ℓ), only a finite subset of walls of marginal stability $\Gamma_{(m,n,\ell)} \subset SL_2(\mathbb{Z})$ contribute to the total single center $1/4-$BPS black hole degeneracy.
This implies that the exact modified degeneracy expression should contain information regarding these contributing walls of marginal stability. We therefore performed the computation of $\Gamma_{(m,n,\ell)}$, and deduced bounds on the finiteness of the contributing walls, including the effects of metamorphosis. The phenomenon of metamorphosis needed that we had to further remove/identify contributions from different walls in this set $\Gamma_{(m,n,\ell)}$. This "identification" was the last step in the formula since one must avoid overcounting the negative discriminant states due to metamorphosis, which means carefully treating further contributions to the sum coming from all walls γ for which $n_\gamma = -1$, $m_\gamma = -1$ and $n_\gamma = m_\gamma = -1$. As a sketch of the final formula, we are able to show that the walls fall into three orbits:

1. The *singlet orbit*. There is no identification on these walls and these walls always have a contribution to the index. The reason that they are singlet orbits is that there is no transformation that takes them into another wall with an equal contribution. They form cycles of unit length.

2. The *doublet orbit*. These orbits consist of cycles of length 2 and occur when either the electric or magnetic charges are -1. For each of the walls in this orbit, there is a $\mathbb{Z}_2$ identification i.e., there exists a metamorphic dual with another wall $\tilde{\gamma}$ in the set of solutions. The only contribution comes from not just a single wall but rather from the each doublet as a whole.

3. Finally, the case where the negative discriminant state has $m = -1, n = -1$ in which case there is $\mathbb{Z}$ orbit. For a wall γ, there are an infinite number of duals $(\tilde{\gamma}_1, \tilde{\gamma}_2, \cdots)$ which is the ∞ orbit. As in the previous case, the contribution comes not from each wall but from each wall in the orbit but from the orbit as a whole. We also show that there is only one cycle and therefore we do not mention it here.

In order to obtain a match with the black hole degeneracies obtained via the localization of the $\mathcal{N} = 4$ quantum entropy function (viz. coefficients of the Bessel functions), we recap the notion that the contribution comes from are different orbits with cycles of length $L = 1$ (singlet), $L = 2$ (doublet) and $L = \infty$ (infinite). For each cycle μ_L in the orbits, we can compute the degeneracy from any representative of the cycle, This gives a coset model for the computation of the black hole entropy where we are able to show that there is only a finite contribution from walls of marginal stability. In fact we also show that the size of these bounds is strongly dependent on the value of the magnetic charge of the original state.

10 Implications of the results

1. We are able to effectively compute the black hole degeneracies for single center black holes. This computation involves the derivation of a new formula which also resolves the discrepancies in previous formula.

2. This formula is highly bounded in the contributions and therefore is more effective existing methods of computation of black hole degeneracies/coefficients of Fourier expansions of Siegel modular forms. It also presents as faster way of extracting these numbers. Much faster.

3. We are able to analytically prove a very surprising result; **the single center $1/4-$BPS black hole degneracies are controlled by multicenter black holes that are $1/2-$BPS bound states.**

11 Comments on the CHL extensions

In this section, we propose extensions to test the theory that we have derived. As it stands, we know now of a fast technique to recover single center $1/4-$BPS black hole degeneracies. An extension of this technique to subgroups of $SL_2(\mathbb{Z})$ should be possible since the modifications to the formula are quite minor but subtle. We may use this to extend the theory put forward in [17] to congruence subgroups of $SL_2(\mathbb{Z})$. We hope for this result to hold since if we are to take exact holography seriously, this should extend to subgroups of $SL_2(\mathbb{Z})$ as well. To do this, we start by looking at the most general example: CHL orbifolds of $\mathcal{N} = 4, d = 4$ black holes, for which the geometric construction is known. In this CHL extension, the first step is the construction of the lift. We outline the key steps for this extension.

1. Step 1: Construct the lift $\Phi_k(\tau, \sigma, z)$ and verify meromorphicity and matching of cusps under a congruence subgroup of $Sp(2, \mathbb{Z})$. This is no longer a Siegel modular form of weight 10 but rather of reduced weight.

2. Depending on how the $\mathbb{Z}_N$ orbifold acts on $Sp(2,\mathbb{Z})$, we need different lifts, which transform under $\Gamma_0(N)$ and $\Gamma^0(N)$.

3. The multiple lifts are required to ensure that the entire pole structure in the SUHP is accounted for. For the non-CHL case, these multiple lifts are trivially identical since $\Gamma_0(1) = \Gamma^0(1) = SL_2(\mathbb{Z})$.

4. For $N \geq 2$, there are inequivalent cusps which need to be summed over. The behaviour of a Siegel lift $\Phi_k(\tau,\sigma,z)$ is such that it doesn't necessarily behave as a modular object for all possible congruence subgroups of $Sp(2,\mathbb{Z})$.

5. We therefore need two lifts $\Phi_k(\tau,\sigma,z)$, $\tilde{\Phi}_k(\tilde{\tau},\tilde{\sigma},\tilde{z})$ which are related by an $Sp(2,\mathbb{Z})$ transform (or by redefinition of charges). Integrality of black hole degeneracies demand in some cases a linear combination of these two lifts, for example when there are dyons with torsion [36][37][38][39].

The lifts are given by [40][32]

$$
\begin{aligned}
\tilde{\Phi}_k(\tilde{\tau},\tilde{\sigma},\tilde{z}) = {} & (-i\sqrt{N})^{-k-2} q^{1/N} y p \\
& \prod_{r=0}^{N-1} \prod_{\substack{(k,\ell,b)>0,\\ k\in\mathbb{Z}+r/N}} \left(1 - q^k y^b p^\ell\right)^{1/2\sum_{s=0}^{N-1} e^{-2\pi i \ell s/N} c^{(r,s)}(\Delta)} \\
& \prod_{r=0}^{N-1} \prod_{\substack{(k,\ell,b)>0,\\ k\in\mathbb{Z}-r/N}} \left(1 - q^k y^b p^\ell\right)^{1/2\sum_{s=0}^{N-1} e^{2\pi i \ell s/N} c^{(r,s)}(\Delta)}
\end{aligned}
$$

where the coefficient in the exponent are defined as in

$$
F^{(r,s)}(\tau,z) = \sum_{n,l} c^{(r,s)}(\Delta) q^n y^\ell
$$

with the definitions

$$
\begin{aligned}
F^{(0,0)}(\tau,z) &= \frac{2}{N}\phi_{0,1}(\tau,z), \\
F^{(0,s)}(\tau,z) &= \frac{2}{N^2+N}\phi_{0,1}(\tau,z) - \frac{2}{N+1}\frac{\vartheta_1^2(\tau,z)}{\eta(\tau)^6} E_N(\tau) \\
F^{(r,rk)}(\tau,z) &= \frac{2}{N^2+N}\phi_{0,1}(\tau,z) - \frac{2}{N^2+N}\frac{\vartheta_1^2(\tau,z)}{\eta(\tau)^6} E_N(\frac{\tau+k}{N})
\end{aligned}
$$

and

$$
E_N(\tau) = 1 + \frac{24}{N-1} \sum_{\substack{n_1,n_2\geq 1 \\ n_1\neq 0 \mod N}} q^{n_1 n_2}.
$$

These are the twisted-twined elliptic genera for $K3$. and therefore for $K3$ sigma model (There are also extensions to the T^4 sigma model [40]). Similarly, the other inequivalent lift is given by

$$\Phi_k(\tau,\sigma,z) = -qyp \prod_{r=0}^{N-1} \prod_{(k,l,b)>0} \left(1 - e^{2\pi i r/N} q^k y^b p^l\right)^{1/2c^{(r,s)}(\Delta)}$$
$$\prod_{r=0}^{N-1} \prod_{(k,l,b)>0} \left(1 - e^{-2\pi i r/N} q^k y^b p^l\right)^{1/2c^{(r,s)}(\Delta)}$$

The behaviour of these functions close to the poles is similar to the case of the $SL_2(\mathbb{Z})$ case in the sense that there is a decay into two eta products. This is expected since we still expect that the $1/4-$BPS black hole decays into two $1/2-$BPS black holes, but under a $\mathbb{Z}_N$ orbifold,

$$\Phi_k(\tau,\sigma,z) \sim 4\pi^2 z^2 \left(\eta(\tau)\eta(N\tau)\right)^{k+2} \left(\eta(\sigma)\eta(N\sigma)\right)^{k+2}$$
$$\tilde{\Phi}_k(\tilde{\tau},\tilde{\sigma},\tilde{z}) \sim (i\sqrt{N})^{-k-2} 4\pi^2 z^2 \left(\eta(\tau)\eta(\tau/N)\right)^{k+2} \left(\eta(\sigma)\eta(N\sigma)\right)^{k+2}$$

The difference here is that $\tilde{\Phi}_k(\tilde{\tau},\tilde{\sigma},\tilde{z})$ is modular on $\Gamma_0(N)$ while $\Phi_k(\tau,\sigma,z)$ is modular on $\Gamma^0(N)$ The next step is to subtract the polar term from the Jacobi forms obtained in the Jacobi-Fourier decomposition of the lifts. For this, we need the Apell-Lerch sum for such models. They can be computed under the appropriate images of the congruent subgroups of $Sp(2,\mathbb{Z})$ can have been computed in [38]. A remark that we would like to make here is that there is a strong reason to look at prime orbifolds for now; this is because the Fourier expansion for prime orbifolds of the lift have integral coefficients. However, the problem to be solved right now is with the extraction of the exact degeneracies via the Rademacher expansion for (some) linear combination $f_1\Phi_k(\tau,\sigma,z) + f_2\tilde{\Phi}_k(\tilde{\tau},\tilde{\sigma},\tilde{z})$. A way forward would be to sum the expressions separately, perform the Rademacher expansions independently, and put them back together. This is the current state of the art from the study of this model to CHL extensions.

Once the Rademacher expansion has been studied, the next step would be to generalize the localization of supergravity set up to these CHL models. Unfortunately, there is no comment to be made here at this point in time.

12 Outlook and future directions

From the framework of this project, there are many vital extensions. The first step is to generalize the machinery to subgroups of the modular group. Physically speaking, this corresponds to the extension to CHL models which are obtained when the Calabi-Yau in the black hole geometry is orbifolded under a fixed-point free action [41]. This is currently being investigated both in terms of the modified black hole formula and in terms of the underlying number theoretic and localization techniques. This is one of the projects that is currently being discussed with members of the SITP, Stanford. Thus, we have expanded a simple proposal to a possible 4-5 paper collaboration.

While the connection between black holes and automorphic forms was investigated to great detain from the perspective of $\mathcal{N} = 4$ theories here, $\mathcal{N} = 2$ theories remain a difficult avenue to investigate. This is again one of the problems being worked on with members of the SITP, Stanford.

As a final statement to end this report, I would like to conclude prohpetically that the relation between number theory, geometry and physics, especially between automorphic forms and physics holds a lot of new avenues of research and new connections between mathematics and physics.

"My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include not only theta-functions but mock theta functions...But before this can happen the purely mathematical exploration of the mock modular forms and their mock-symmetries must be carried a great deal further." - Freeman Dyson (1987)

13 Academic continuation, expectations versus achievements

As mentioned in , there has been a continued exchange of ideas and collaboration between Shamit Kachru's group and myself following this scientific visit. I believe this is highly beneficial to the TU Wien to have academic exchanges with a mathematical physicist of the calibre of Shamit Kachru and his group. This continuation was one of the main goals of my proposal.
Another goal which I hoped to achieve was that of participation in various seminars, workshop and conferences and this too has been achieved during my stay there.
Overall, I am pleased to report that in lieu of the imminent publication of results, I have fulfilled all the targeted goals within the scope of the Marshall fellowship proposal.

These achievements would have not been possible without the Austrian Marshall Plan Fellowship and I am grateful for having received the scholarship.

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