

## FINAL REPORT

### *C<sup>0</sup>-(IN)EXTENDIBILITY IN MATHEMATICAL GENERAL RELATIVITY*

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#### OVERVIEW

This report consists of the two papers that resulted from my stay at the University of Miami thanks to the scholarship of the Austrian Marshall Plan foundation. The first is already published and the second has been submitted.

- (1) Graf, M., Ling, E., Maximizers in Lipschitz spacetimes are either timelike or null, *Class. Quantum Grav.*, <https://doi.org/10.1088/1361-6382/aab259>, 2018
- (2) Galloway, G.J., Graf, M., Rigidity of asymptotically  $AdS_2 \times S^2$  spacetimes, 2016. Preprint: <https://arxiv.org/pdf/1609.04939>,

## Maximizers in Lipschitz spacetimes are either timelike or null

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March 6, 2018

### Abstract

We prove that causal maximizers in  $C^{0,1}$  spacetimes are either timelike or null. This question was posed in [17] since bubbling regions in  $C^{0,\alpha}$  spacetimes ( $\alpha < 1$ ) can produce causal maximizers that contain a segment which is timelike and a segment which is null, cf. [3]. While  $C^{0,1}$  spacetimes do not produce bubbling regions, the causal character of maximizers for spacetimes with regularity at least  $C^{0,1}$  but less than  $C^{1,1}$  was unknown until now. As an application we show that timelike geodesically complete spacetimes are  $C^{0,1}$ -inextendible.

## 1 Introduction

Recently, there has been an interest in low regularity aspects of general relativity motivated in part by the *strong cosmic censorship conjecture*. Roughly, the conjecture states that the maximal globally hyperbolic development of generic initial data for the Einstein equations is inextendible as a suitably regular Lorentzian manifold. Formulating a precise statement of the strong cosmic censorship conjecture is itself a challenge because one needs to make precise the phrases ‘generic initial data’ and ‘suitably regular Lorentzian manifold’. Understanding the latter is where general relativity in low regularity and in particular (in-)extendibility results become significant.

Christodoulou [1] established a  $C^0$ -inextendibility result for spacetimes satisfying the Einstein-scalar field equations within the class of spherically symmetric spacetimes. Likewise, the authors in [7] demonstrated the  $C^0$ -inextendibility of open FLRW spacetimes which are not Milne-like also within the class of spherically symmetric spacetimes. Moreover, they demonstrate, to the contrary, that Milne-like spacetimes are always  $C^0$ -extendible (but might be  $C^2$ -inextendible). Dafermos [4, 5] demonstrated the  $C^0$ -extendibility of the maximal globally hyperbolic development of solutions to the spherically symmetric Einstein-Maxwell-scalar field system arising from small perturbations of Reissner-Nordström initial data. In fact, more recently, Dafermos and Luk [6] have given a proof, without symmetry assumptions, of the  $C^0$  stability of the Kerr-Cauchy horizon. This gives firm evidence that the strong cosmic censorship conjecture is false in the  $C^0$  setting. The current suggestion for the statement of

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the strong cosmic censorship conjecture is to require inextendibility as a Lorentzian manifold with a continuous metric and Christoffel symbols locally in  $L^2$  [2, 6].

Still, there are some interesting new results establishing the  $C^0$ -inextendibility of spacetimes without any symmetry assumptions of the extension. The first such example was given by Sbierski [18] with his proof of the  $C^0$ -inextendibility of the Schwarzschild spacetime. Further advancements have been made in this direction [8], and our second application adds to the list of these inextendibility results.

A systematic study of general relativity in low regularity began with the influential paper [3] where the authors studied causal theory in spacetimes where the regularity of the metric was less than  $C^2$ . Since then further advancements have established that most of classical causal theory remains valid for  $C^{1,1}$  spacetimes [11, 12, 16] and even the singularity theorems hold in this regularity class [13, 14, 9]. Therefore spacetimes with  $C^{1,1}$  metrics can be seen as the threshold to where classical causality theory applies.

Once the regularity drops below  $C^{1,1}$ , causality theory departs significantly from classical theory. Example 1.11 in [3] shows that the push-up Lemma (i.e.  $I^+(J^+(\Omega)) = I^+(\Omega)$ ) does not necessarily hold in  $C^{0,\alpha}$  spacetimes,  $\alpha \in (0, 1)$ . This led the authors to define the so-called bubbling regions for these spacetimes. From Example 1.11 it is readily seen that any causal curve from the origin to a point  $p$  in the bubbling region must begin null on some interval. Thus any causal curve which maximizes the Lorentzian length between the origin and  $p$  is null on an interval, hence not timelike. However, since these points have positive Lorentzian distance, it must be timelike on a set of non-zero measure, so it cannot be null. This deviates drastically from classical (at least  $C^{1,1}$ ) theory where maximizers must be geodesics and hence either timelike or null.

On the other hand [3, Corollary 1.17] shows that  $C^{0,1}$  spacetimes (i.e. spacetimes with a Lipschitz continuous metric) do not admit bubbling regions. This leaves open the question of whether maximizers in  $C^{0,1}$  spacetimes must be either timelike or null. In fact this question was posed in [17]. In this paper we prove the following theorem which answers the question affirmatively.

**Theorem 1.1.** *Let  $(M, g)$  be a Lipschitz spacetime. If  $p, q \in M$  with  $q \in J^+(p)$ , then any maximizing causal curve from  $p$  to  $q$  is either timelike or null.*

As an application we show

**Theorem 1.2.** *Let  $(M, g)$  be a smooth timelike geodesically complete spacetime. Then  $(M, g)$  is  $C^{0,1}$ -inextendible.*

This provides a partial answer to a question raised in [18] which asks whether timelike geodesically complete spacetimes are  $C^0$ -inextendible. Together with the corresponding result from [8], which states that smooth timelike geodesically complete globally hyperbolic spacetimes are  $C^0$ -inextendible, this shows that if  $(M_{\text{ext}}, g_{\text{ext}})$  is a  $C^0$  extension of a timelike geodesically complete spacetime  $(M, g)$ , then  $(M, g)$  is not globally hyperbolic and  $g_{\text{ext}}$  is not Lipschitz.

We also note that this establishes that strongly causal Lipschitz spacetimes are examples of *regular* Lorentzian length spaces as introduced in [10].

## 2 Proofs of Theorems 1.1 and 1.2

Our definitions for a  $C^{k,\alpha}$  spacetime and future causal curves  $\gamma: I \rightarrow M$  follow that of [3]. In particular we use Lipschitz causal curves which implies that the derivative  $\dot{\gamma}$  of  $\gamma$  exists almost everywhere and  $\dot{\gamma}$  is locally in  $L^\infty$ . We say a Lipschitz curve is future causal if  $\dot{\gamma}$  is future causal almost everywhere. A future causal curve is *timelike* if  $\dot{\gamma}$  is timelike almost everywhere and it is *null* if  $\dot{\gamma}$  is null almost everywhere. The *Lorentzian length* of a future causal curve  $\gamma: I \rightarrow M$  is the integral  $L(\gamma) = \int_I \sqrt{-g(\dot{\gamma}, \dot{\gamma})}$  and  $\gamma$  is said to be *maximizing* or a *maximizer* if  $L(\gamma) \geq L(\lambda)$  for any future causal curve  $\lambda$  whose endpoints agree with those of  $\gamma$ .

*Proof of Theorem 1.1.* Suppose  $\gamma: I \rightarrow M$  is a maximizing future directed causal curve from  $p$  to  $q$  which is not null. We will prove that  $\gamma$  is timelike. Seeking a contradiction, suppose the set  $N_I := \{s \in I : \dot{\gamma}(s) \text{ exists and is null}\} \subseteq I$  has positive measure. Below we will construct another causal curve from  $p$  to  $q$  which is longer than  $\gamma$ , and hence contradicting the fact that  $\gamma$  is a maximizer.

To do this, we first want to localize the situation. By compactness we may cover  $I$  by finitely many open intervals  $I_k$  (half open intervals on the endpoints of  $I$ ) such that each  $\gamma|_{I_k}$  is contained in a relatively compact chart domain  $(U_k, \varphi_k)$  on which  $g(\partial_0^{\varphi_k}, \partial_0^{\varphi_k}) < c_k < 0$ . We are now going to show that for at least one of these  $I_k$ , we have  $0 < \mu(N_{I_k}) < \mu(I_k)$  where  $\mu$  is the usual Lebesgue measure on  $\mathbb{R}$ . That is we will show there is a  $k$  such that  $\gamma|_{I_k}$  is causal but neither timelike nor null. Assume for the moment that  $\mu(N_{I_j}) = 0$  for some  $j$ . Then, since the intersection of the neighbouring intervals  $I_{j-1}$  and  $I_{j+1}$  with  $I_j$  must be non-empty and open, either one of those has the desired property or  $\mu(N_{I_{j-1}}) = \mu(N_{I_{j+1}}) = 0$ . The existence of a suitable  $I_k$  now follows by induction and noting that  $\mu(N_I) \neq 0$ . If instead  $\mu(N_{I_j}) = \mu(I_j)$ , one proceeds the same way, using  $\mu(N_I) \neq \mu(I)$  in the end.

This shows that we may assume w.l.o.g. that  $\gamma(I)$  is contained in such a chart domain  $(U, \varphi)$ . By reparametrizing  $\gamma$  we may further assume that  $\dot{\gamma}(0)$  exists and is timelike. Using a linear change of coordinates corresponding to a Gram-Schmidt orthogonalization process of  $\{\partial_0^\varphi, \dots, \partial_{n-1}^\varphi\}|_{\gamma(0)}$  and a translation we get new coordinates  $\psi$  on  $U$  for which  $\partial_0^\psi \propto \partial_0^\varphi$  (and hence  $g(\partial_0^\psi, \partial_0^\psi) < c < 0$ ) on  $U$ ,  $\gamma^\psi(0) = 0$  and  $g^{\psi}(0) = \eta$  where  $\eta$  is the Minkowski metric.

To sum up, we need only consider the case  $M = \mathbb{R}^n$ ,  $\gamma: [a, b] \rightarrow U \subseteq \mathbb{R}^n$ ,  $\gamma(0) = 0$ ,  $\gamma$  is differentiable at 0 and  $\dot{\gamma}(0)$  is timelike,  $\mu(N_{[a,0]}) > 0$  (if instead  $\mu(N_{[0,b]}) > 0$  one just needs to reverse the time orientation),  $g(0) = \eta$  and  $\partial_0$  (uniformly) timelike on  $U$ .

We first look at  $\gamma_1 := \gamma|_{[a,0]}$ . As in [3, Lem. 1.15], given a Lipschitz function  $f: [a, 0] \rightarrow [0, \infty)$ , we define a new Lipschitz curve  $\Gamma_1: [a, 0] \rightarrow \mathbb{R}^n$  by  $\Gamma_1^\mu(s) := \gamma_1^\mu(s) + \epsilon f(s)T^\mu$  where  $T^\mu$  is defined via  $\partial_0 = T^\mu \partial_\mu$  (i.e.  $T^0 = 1$  and  $T^\mu = 0$  for  $\mu \neq 0$ ). Then the proof of [3, Lem. 1.15] shows that for a Lipschitz metric one can find a specific  $f \in C^{0,1}([a, 0])$  such that  $\Gamma_1 \subseteq U$ ,  $\Gamma_1(a) = \gamma_1(a)$ ,  $\Gamma_1(0) = (\epsilon f(0), 0, \dots, 0)$  and

$$g_{\Gamma_1}(\dot{\Gamma}_1, \dot{\Gamma}_1) \leq g_{\gamma_1}(\dot{\gamma}_1, \dot{\gamma}_1) - \frac{\epsilon}{2} \quad (2.1)$$

a.e. on  $[a, 0]$  for all  $\epsilon$  less than some  $\epsilon_0$ . This gives the following estimate for the lengths of

$\gamma_1$  and  $\Gamma_1$ :

$$L(\Gamma_1) = \int_a^0 \sqrt{-g_{\Gamma_1}(\dot{\Gamma}_1, \dot{\Gamma}_1)} = \int_{N_{[a,0]}} \sqrt{-g_{\Gamma_1}(\dot{\Gamma}_1, \dot{\Gamma}_1)} + \int_{[a,0] \setminus N_{[a,0]}} \sqrt{-g_{\Gamma_1}(\dot{\Gamma}_1, \dot{\Gamma}_1)} \quad (2.2)$$

$$\geq \frac{\mu(N_{[a,0]})}{\sqrt{2}} \sqrt{\varepsilon} + \int_{[a,0] \setminus N_{[a,0]}} \sqrt{-g_{\Gamma_1}(\dot{\Gamma}_1, \dot{\Gamma}_1)} \quad (2.3)$$

$$\geq \frac{\mu(N_{[a,0]})}{\sqrt{2}} \sqrt{\varepsilon} + L(\gamma_1). \quad (2.4)$$

Note that this not only shows that  $L(\Gamma_1) \geq L(\gamma_1)$  but more importantly that the length difference is bounded from below by an expression that scales like  $\sqrt{\varepsilon}$  as  $\varepsilon \rightarrow 0$ .

We now turn to  $\gamma|_{[0,b]}$  and assume that  $\gamma|_{[0,b]}$  is parametrized by the  $x^0$ -coordinate. The chain rule ensures that  $\dot{\gamma}(0)$  exists under this reparameterization and hence remains timelike. Since  $\Gamma_1(0) \neq \gamma(0)$  we have to find  $\tau_\varepsilon > 0$  and a future directed causal curve  $\Gamma_2$  from  $\Gamma_1(0) = (\varepsilon f(0), 0)$  to  $\gamma(\tau_\varepsilon) = (\tau_\varepsilon, \bar{\gamma}(\tau_\varepsilon)) \in \mathbb{R} \times \mathbb{R}^{n-1}$  (see figure 1) and such that for  $\varepsilon$  small enough  $L(\Gamma_1) + L(\Gamma_2) > L(\gamma_1) + L(\gamma_2)$  where  $\gamma_2 := \gamma|_{[0,\tau_\varepsilon]}$ . It should be noted that the segment  $\gamma_2$  of  $\gamma$  depends on  $\tau_\varepsilon$  and hence depends on  $\varepsilon$  itself. It suffices to show  $L(\Gamma_1) > L(\gamma_1) + L(\gamma_2)$ . And using (2.2) we see that this holds if  $L(\gamma_2)$  has an upper bound that scales like  $\varepsilon^k$  for some  $k > \frac{1}{2}$  as  $\varepsilon \rightarrow 0$ . Below we will show this is true for  $k = 1$ .

Since  $\gamma$  is differentiable at  $s = 0$ , Taylor's theorem gives

$$\gamma(s) = \gamma(0) + \dot{\gamma}(0)s + h(s)s = (\dot{\gamma}(0) + h(s))s, \quad (2.5)$$

where  $h(s) \rightarrow 0$  as  $s \rightarrow 0$ . Since we further assumed  $\dot{\gamma}(0) =: (v^0, \bar{v})$  to be timelike and  $g(0) = \eta$ , we can choose  $\alpha > 1$  such that  $\frac{v^0}{|\bar{v}|_e} > \alpha$  where  $|\cdot|_e$  is the usual Euclidean norm. Together with (2.5) this shows that there exists  $s_0 > 0$  such that  $s = \gamma^0(s) > \alpha|\bar{\gamma}(s)|_e$  for all  $s < s_0$ . Choose  $1 < \beta < \alpha$  and let  $C_{\beta,d}$  denote the future cone with slope  $\beta$  that has its tip in  $(d, 0)$ , i.e.,  $C_{\beta,d} = \{(\beta|\bar{x}|_e + d, \bar{x}) : \bar{x} \in \mathbb{R}^{n-1}\}$ . Since  $\beta > 1$  and  $g(0) = \eta$  the continuity of  $g$  allows us to find a small neighbourhood  $V$  of 0 such that  $C_{\beta,d} \cap V \setminus \{(d, 0)\} \subseteq I^+((d, 0))$  for all small  $d > 0$ . We have  $(s, \bar{\gamma}(s)) \in C_{\beta, s - \beta|\bar{\gamma}(s)|_e} \cap V \subseteq I^+((s - \beta|\bar{\gamma}(s)|_e, 0))$  for small  $s$ . Since  $s > \alpha|\bar{\gamma}(s)|_e$  we have  $s - \beta|\bar{\gamma}(s)|_e > s(1 - \frac{\beta}{\alpha})$  which shows that  $(s, \bar{\gamma}(s)) \in I^+((s(1 - \frac{\beta}{\alpha}), 0))$  for small  $s$ . Thus  $\gamma(\tau_\varepsilon) = (\tau_\varepsilon, \bar{\gamma}(\tau_\varepsilon)) \in I^+((\varepsilon f(0), 0))$  for  $\tau_\varepsilon = \frac{f(0)}{1 - \frac{\beta}{\alpha}} \varepsilon$  if  $\varepsilon$  is small enough. Therefore we have demonstrated that we can construct  $\Gamma_2$  from  $\Gamma_1(0)$  to  $\gamma(\tau_\varepsilon)$  for small  $\varepsilon$ . Finally, for  $L(\gamma_2)$  an estimate from the proof of Thm. 3.3 in [8] shows that, if  $\gamma_2 \subseteq V$ , where  $V$  is a neighborhood of 0 on which  $|g_{\mu\nu} - \eta_{\mu\nu}| < \delta$ , then

$$L(\gamma_2) \leq \tau_\varepsilon \sqrt{1 + \delta + 4(n-1)^2 \delta} = \varepsilon \frac{f(0)}{1 - \frac{\beta}{\alpha}} \sqrt{1 + \delta + 4(n-1)^2 \delta}. \quad (2.6)$$

So  $L(\gamma_2)$  is indeed bounded from above by a term of order  $\varepsilon$ . Now choose  $\varepsilon$  small enough so that

$$\frac{\mu(N_{[a,0]})}{\sqrt{2}} \sqrt{\varepsilon} \geq \varepsilon \frac{f(0)}{1 - \frac{\beta}{\alpha}} \sqrt{1 + \delta + 4(n-1)^2 \delta},$$

then we have

$$\begin{aligned}
 L(\Gamma_1) + L(\Gamma_2) &\geq L(\Gamma_1) \geq L(\gamma_1) + \frac{\mu(N_{[a,0]})}{\sqrt{2}} \sqrt{\varepsilon} \\
 &\geq L(\gamma_1) + \varepsilon \frac{f(0)}{1 - \frac{\beta}{\alpha}} \sqrt{1 + \delta + 4(n-1)^2 \delta} \\
 &\geq L(\gamma_1) + L(\gamma_2),
 \end{aligned}$$

and so we are done.  $\square$

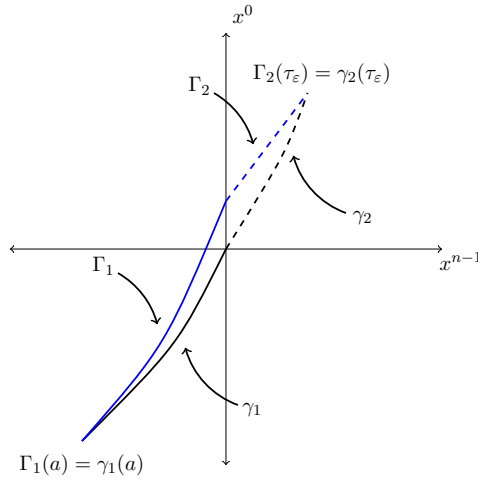


Figure 1: The causal curve formed by concatenating  $\Gamma_1$  and  $\Gamma_2$  has Lorentzian length greater than that of  $\gamma|_{[a, \tau_\varepsilon]}$ . Hence  $\gamma$  cannot be maximizing.

*Remark.* Our proof also shows if  $\gamma$  is a maximizing null curve between two points, then  $\gamma$  does not contain a single timelike tangent. This result is also obtained in [15, Theorem 18] under much weaker differentiability assumptions.

*Proof of Theorem 1.2.* Seeking a contradiction, suppose such a Lipschitz extension  $(M_{\text{ext}}, g_{\text{ext}})$  exists. Following the proof of [8, Theorem 3.3], the future causal curve  $\alpha$  is a maximizer from  $q \in M$  to  $p \in \partial^+ M$  within a small globally hyperbolic set  $V$  containing  $q$  and  $p$ . Since  $q$  and  $p$  are timelike separated within  $V$ , the curve  $\alpha$  is timelike by Theorem 1.1. Since  $\alpha$  is a timelike maximizer and leaves  $M$ , the portion of  $\alpha$  that lies in  $M$  is an inextendible (within  $M$ ) timelike geodesic. Then timelike geodesic completeness of  $(M, g)$  implies  $L(\alpha) = \infty$ , contradicting  $L(\alpha) = d_V(q, p) < \infty$ .  $\square$

Lastly we remark that in [17], it was suggested that if maximizers in  $C^1$  spacetimes are timelike, then it may be possible to apply the du Bois-Reymond-trick to show that the maximizers must be  $C^2$  timelike geodesics. However it seems like the du Bois-Reymond-trick does not apply for timelike maximizers in the Lorentzian setting. This is because one can not ensure that the variation of the maximizer is timelike, since the tangent of a Lipschitz timelike

curve may still come arbitrarily close to being null. This is not an issue in the Riemannian setting since there is no causal distinction between the curves.

*Acknowledgements.* Melanie Graf is the recipient of a DOC-fellowship of the Austrian Academy of Sciences. This work was completed as part of her research stay at the University of Miami made possible by a scholarship of the Austrian Marshall Plan Foundation. This work was also partially supported by project P28770 of the Austrian Science Fund FWF. The authors are grateful to Greg Galloway for helpful comments. The authors would also like to thank Clemens Sämann and Michael Kunzinger for suggesting the problem and Ettore Minguzzi for bringing [15] to their attention. We also thank an anonymous referee for pointing out the problems with the du Bois-Reymond-trick.

## References

- [1] Christodoulou, D., The instability of naked singularities in the gravitational collapse of a scalar field, *Ann. of Math. (2)* **149** (1999), no. 1, 183–217.
- [2] Christodoulou, D., The formation of black holes in general relativity. *EMS Monographs in Mathematics*. European Mathematical Society (EMS), Zürich, 2009.
- [3] Chruściel, P.T., Grant, J.D.E., On Lorentzian causality with continuous metrics, *Classical Quantum Gravity* 29(14) 145001, 32 pp., 2012.
- [4] Dafermos, M., Stability and instability of the Cauchy horizon for the spherically symmetric Einstein-Maxwell-scalar field equations, *Ann. of Math. (2)* **158** (2003), no. 3, 875–928.
- [5] Dafermos, M. The interior of charged black holes and the problem of uniqueness in general relativity, *Comm. Pure Appl. Math.* **58** (2005), no. 4, 445–504.
- [6] Dafermos, M., Luk, J., The interior of dynamical vacuum black holes I: The  $C^0$ -stability of the Kerr Cauchy Horizon, *arXiv:1710.01722*.
- [7] Galloway, G., Ling, E., Some Remarks on the  $C^0$ -(In)Extendibility of Spacetimes. *Ann. Henri Poincaré* 18, pp 3427–3447, 2017.
- [8] Galloway, G., Ling, E., Sbierski, J., Timelike completeness as an obstruction to  $C^0$ -extensions. *Commun. Math. Phys.*, <https://doi.org/10.1007/s00220-017-3019-2>, 2017.
- [9] Graf, M., Grant, J.D.E., Kunzinger, M., Steinbauer, R., The Hawking-Penrose singularity theorem for  $C^{1,1}$ -Lorentzian metrics. *Commun. Math. Phys.*, <https://doi.org/10.1007/s00220-017-3047-y>, 2017.
- [10] Kunzinger, M., Sämann, S., Lorentzian length spaces. *arXiv:1711.08990*.
- [11] Kunzinger, M., Steinbauer, R., Stojković, M., The exponential map of a  $C^{1,1}$ -metric. *Differential Geom. Appl.* 34, 14–24, 2014.
- [12] Kunzinger, M., Steinbauer, R., Stojković, M., Vickers, J.A., A regularisation approach to causality theory for  $C^{1,1}$ -Lorentzian metrics. *Gen. Relativ. Gravit.* 46:1738, 18 pp, 2014.

- [13] Kunzinger, M., Steinbauer, R., Stojković, M., Vickers, J.A., Hawking's singularity theorem for  $C^{1,1}$ -metrics. *Classical Quantum Gravity* 32, 075012, 19pp, 2015
- [14] Kunzinger, M., Steinbauer, R., Vickers, J.A., The Penrose singularity theorem in regularity  $C^{1,1}$ . *Classical Quantum Gravity* 32, 155010, 12pp, 2015
- [15] Minguzzi, E., Causality theory for closed cone structures with applications. *arXiv:1709.06494v3*
- [16] Minguzzi, E., Convex neighborhoods for Lipschitz connections and sprays. *Monatsh. Math.* 177, no. 4, 569–625, 2015
- [17] Sämann, C., Steinbauer, R., On geodesics in low regularity. *Journal of Physics: Conferences Series*. 968, 2018
- [18] Sbierski, J., The  $C^0$ -inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian geometry. Preprint *arXiv:1507.00601*. To appear in *Journal of Differential Geometry*.



Rigidity of asymptotically  $AdS_2 \times S^2$  spacetimesGregory J. Galloway<sup>\*</sup> and Melanie Graf<sup>†</sup><sup>\*</sup>Department of Mathematics,  
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April 1, 2018

**Abstract**

The spacetime  $AdS_2 \times S^2$  is well known to arise as the ‘near horizon’ geometry of the extremal Reissner-Nordstrom solution, and for that reason it has been studied in connection with the AdS/CFT correspondence. Here we consider asymptotically  $AdS_2 \times S^2$  spacetimes that obey the null energy condition (or a certain averaged version thereof). In support of a conjecture of Juan Maldacena, we show that any such spacetime must have a geometry very similar to  $AdS_2 \times S^2$  (and under certain circumstances must be isometric to  $AdS_2 \times S^2$ ).

**1 Introduction**

An interesting feature of the spacetime  $AdS_2 \times S^2$  is that it arises as the ‘near horizon’ geometry of the extremal Reissner-Nordstrom solution; see e.g. [6]. For this reason, this spacetime (sometimes referred to as the Robinson-Bertotti solution) has been studied in various works in connection with the AdS/CFT correspondence [8]; see e.g. [10] and references therein. More recently a class of horizon free supersymmetric solutions to Einstein-Maxwell theory having  $AdS_2 \times S^2$  asymptotics has been constructed by Lunin [7]. However, on the basis of an example considered in [10, Section 2.2], and also a result in [5] (Theorem 2.1), Maldacena has suggested that any asymptotically  $AdS_2 \times S^2$  spacetime that obeys the null energy condition (NEC), or more generally the average null energy condition (ANEC), should be quite special. In fact he has conjectured that the only such spacetime is  $AdS_2 \times S^2$  itself [9]. In particular, consistent with the example in [10] mentioned above, 4-dimensional spacetimes that satisfy the ANEC *strictly* could not have  $AdS_2 \times S^2$  asymptotics. All of this suggests

that the examples constructed in [7] cannot be globally regular, which, in fact, has since been confirmed by Lunin [9].

In this paper we obtain some results on the *rigidity* of asymptotically  $AdS_2 \times S^2$  spacetimes satisfying the NEC, which support the conjectural picture put forth by Maldacena. While precise statements are postponed to Section 3, our main result may be paraphrased as follows.

**Theorem 1.1.** *Let  $(M, g)$  be an asymptotically  $AdS_2 \times S^2$  spacetime (see Definition 2.4) that satisfies the null energy condition (NEC),  $\text{Ric}(X, X) \geq 0$  for all null vectors  $X$ . Then the following holds.*

- (i)  $(M, g)$  is foliated by smooth totally geodesic null hypersurfaces  $N_u \approx \mathbb{R} \times S^2$ ,  $u \in \mathbb{R}$ .
- (ii) By time-dualizing, one obtains a second foliation by smooth totally geodesic null hypersurfaces  $\tilde{N}_v \approx \mathbb{R} \times S^2$ ,  $v \in \mathbb{R}$ , transverse to the foliation  $\{N_u\}_{u \in \mathbb{R}}$ . By considering the intersections of the  $N_u$ 's and  $\tilde{N}_v$ 's, this double null foliation gives rise to a foliation of  $(M, g)$  by totally geodesic isometric round (i.e. constant curvature) 2-spheres  $S(u, v)$ .

The properties (i) and (ii) are, of course, basic features of  $AdS_2 \times S^2$ . One of the main results leading to the proof of Theorem 1.1, Proposition 3.3, together with a known result concerning the existence of null conjugate points [14, 2] confirms that there do not exist any asymptotically  $AdS_2 \times S^2$  spacetimes obeying the strict ANEC.

While we have stated Theorem 1.1 and Theorem 1.2 below with respect to the NEC, in fact both results remain valid under a weaker curvature condition (which, however, is stronger than the ANEC): It is sufficient to assume that along all future or past complete null rays  $\eta : [0, \infty) \rightarrow M$ , one has,

$$\int_0^\infty \text{Ric}(\eta'(s), \eta'(s)) ds \geq 0. \quad (1.1)$$

In order to simplify a bit the presentation of the proofs of Theorems 1.1 and 1.2, we postpone to an appendix a discussion of the changes needed to prove these theorems under the curvature condition (1.1).

Theorem 1.1 falls short of showing that  $(M, g)$  splits as a metric product along the totally geodesic 2-spheres. A necessary condition for this is that the distribution of timelike 2-planes orthogonal to the 2-spheres be integrable. This is what occurs in the next result. Although not itself an Einstein manifold,  $AdS_2 \times S^2$  is a product of Einstein manifolds, and as such its Ricci tensor is covariant constant,  $\nabla \text{Ric} = 0$ . Under this added assumption we obtain the following.

**Theorem 1.2.** *Let  $(M, g)$  be an asymptotically  $AdS_2 \times S^2$  spacetime that satisfies the NEC. If the Ricci tensor is covariant constant then  $(M, g)$  is globally isometric to  $AdS_2 \times S^2$ .*

We would like to say a word about the approach to the asymptotics taken here. One possible approach to the asymptotics, which will be considered in a subsequent paper, is to introduce a notion of a ‘singular’ timelike conformal boundary. In fact,  $AdS_2 \times S^2$  admits, in a fairly natural way, such a boundary. The more customary analytic approach taken in the present paper, is to require that the spacetime metric  $g$  asymptote at a suitable rate, with respect to a natural coordinate system, to the  $AdS_2 \times S^2$  metric  $\mathring{g}$  on approach to infinity. This approach to the asymptotics gives strong control over the causal structure and allows one to obtain rather fine geometric properties needed to establish Theorem 1.1.

In Section 2 we give the formal definition of an asymptotically  $AdS_2 \times S^2$  spacetime, and derive some consequences of the assumed asymptotics. In Section 3 we establish the existence of a foliation by totally geodesic null hypersurfaces, and a foliation by totally geodesic isometric round 2-spheres, thereby establishing Theorem 1.1. In Section 4 we present a proof of Theorem 1.2.

ACKNOWLEDGEMENTS. The authors are very grateful to Juan Maldacena for bringing this problem to their attention and for many valuable comments. The authors would also like to thank Eric Ling for his interest in this work and for helpful comments. GJG’s research was partially supported by the NSF under the grant DMS-171080. MG’s research was supported by project P28770 of the Austrian Science Fund FWF and a scholarship from the Austrian Marshall Plan Foundation to visit the University of Miami.

## 2 Asymptotically $AdS_2 \times S^2$ spacetimes

In this section we describe in a precise manner what it means for a spacetime  $(M, g)$  to be asymptotically  $AdS_2 \times S^2$ , and we obtain some consequences of these assumed asymptotics.

### 2.1 Exact $AdS_2 \times S^2$ space

Let  $\mathring{M} = \mathbb{R} \times \mathbb{R} \times S^2$ . We set

$$\mathring{g} = -\cosh^2(x)dt^2 + dx^2 + d\Omega^2.$$

For future reference, the non-zero Christoffels for this metric are

$$\mathring{\Gamma}_{tx}^t = \tanh(x), \mathring{\Gamma}_{tt}^x = \cosh(x) \sinh(x), \mathring{\Gamma}_{\phi\theta}^\phi = \cot(\theta), \mathring{\Gamma}_{\phi\phi}^\theta = \sin(\theta) \cos(\theta),$$

the Riemann tensor can be expressed as

$$\mathring{R} = \mathring{R}_{AdS_2} + \mathring{R}_{S^2}$$

and the same holds for  $\mathring{\text{Ric}}$ . Explicitly one has

$$\mathring{\text{Ric}}_{tt} = \cosh(x)^2, \mathring{\text{Ric}}_{xx} = -1, \mathring{\text{Ric}}_{\theta\theta} = 1, \mathring{\text{Ric}}_{\phi\phi} = \sin^2(\theta).$$

The scalar curvature vanishes. Note that while  $AdS_2 \times S^2$  is not an Einstein manifold, one can still nicely express  $\mathring{\text{Ric}}$  in terms of the metric:  $\mathring{\text{Ric}} = -\mathring{g} + 2d\Omega^2$ .

## 2.2 The metrics $\mathring{g}_\alpha$

To get a better handle on the asymptotics of  $g$  we will further define a family of metrics  $\mathring{g}_\alpha$  ( $\alpha \in \mathbb{R}_+$ ) on  $\mathring{M}$  via

$$\mathring{g}_\alpha = -\alpha \cosh^2(x) dt^2 + dx^2 + d\Omega^2.$$

The importance of these metrics for the asymptotics lies in Lemma 2.6, stating that, in essence, there exist  $\beta > 1$  and  $\alpha < 1$  such that far out  $\mathring{g}_\alpha \prec g \prec \mathring{g}_\beta$  and  $\beta, \alpha \rightarrow 1$  as one approaches infinity. (Recall, for Lorentzian metrics  $g_1$  and  $g_2$ ,  $g_1 \prec g_2$  means that the null cones of  $g_2$  are wider than those of  $g_1$  in the sense that for any vector  $X \neq 0$ , if  $g_1(X, X) \leq 0$  then  $g_2(X, X) < 0$ ).

Along the null curves of  $\mathring{g}_\alpha$  with  $\theta = \theta_0$ ,  $\phi = \phi_0$ , one has

$$dt = \pm \frac{1}{\sqrt{\alpha} \cosh x} dx.$$

Integrating gives the following.

**Lemma 2.1** (Null curves for  $\mathring{g}_\alpha$ ). *The curves  $s \mapsto (f_\alpha(s, t_0, x_0), s + x_0, \phi_0, \theta_0)$  and  $s \mapsto (-f_\alpha(s, -t_0, x_0), s + x_0, \phi_0, \theta_0)$ , where*

$$f_\alpha(s, t_0, x_0) := \frac{2}{\sqrt{\alpha}} (\tan^{-1}(e^{s+x_0}) - \tan^{-1}(e^{x_0})) + t_0,$$

*are future, resp. past, directed achronal null curves in  $(\mathring{M}, \mathring{g}_\alpha)$  passing through the point  $(t_0, x_0, \phi_0, \theta_0)$ .*

*Remark 2.2.* For future reference we note the following.

$$\lim_{s \rightarrow -\infty} f_\alpha(s, t_0, x_0) = t_0 - \frac{2 \tan^{-1}(e^{x_0})}{\sqrt{\alpha}}$$

and

$$\lim_{s \rightarrow \infty} f_\alpha(s, t_0, x_0) = t_0 - \frac{2 \tan^{-1}(e^{x_0})}{\sqrt{\alpha}} + \frac{\pi}{\sqrt{\alpha}} \leq t_0 + \frac{\pi}{\sqrt{\alpha}}.$$

Since  $f_\alpha$  is increasing this means that  $f_\alpha(s, t_0, x_0) \leq t_0 - \frac{2 \tan^{-1}(e^{x_0})}{\sqrt{\alpha}} + \frac{\pi}{\sqrt{\alpha}}$  for all  $s$ .

The timelike futures  $\mathring{I}_\alpha^+(p)$  are easily seen to satisfy the following.

**Lemma 2.3.** *Let  $p = (t_0, x_0, \omega_0) \in (\mathring{M}, \mathring{g}_\alpha)$ . Then  $\{t > t_0 + \frac{\pi}{\sqrt{\alpha}}\} \times \mathbb{R} \times \{\omega_0\} \subseteq \mathring{I}_\alpha^+(p)$ .*

### 2.3 Definition of asymptotically $AdS_2 \times S^2$ spacetimes

Throughout we shall assume that spacetime is *causally simple*. Following [11, Sec. 3.10], we say that a spacetime is causally simple provided  $J^\pm(p)$  is closed for all  $p \in M$  and  $(M, g)$  is causal (i.e. contains no closed causal curves). As a consequence, the sets  $J^\pm(K)$  are closed for all compact sets  $K$  in  $M$ , and  $(M, g)$  is strongly causal.

In order to prove our main results, a careful treatment of the asymptotics, as layed out in the following definition, is required.

**Definition 2.4.** Let  $(M, g)$  be a 4-dimensional causally simple spacetime. We say that  $(M, g)$  is asymptotically  $AdS_2 \times S^2$  provided the following conditions hold.

- (a<sub>1</sub>) There exists a closed subset  $A \subseteq M$  such that  $M \setminus A^\circ$  is the disjoint union of two manifolds with boundary  $M_1$  and  $M_2$  such that

$$M_1 \cong \mathbb{R} \times (-\infty, -a] \times S^2 \quad \text{and} \quad M_2 \cong \mathbb{R} \times [a, \infty) \times S^2, \\ p \in M_1 \cup M_2 \mapsto (t(p), x(p), \omega(p)),$$

$a \geq 1$ , and the boundary  $\partial A$  is mapped to  $(\mathbb{R} \times \{-a\} \times S^2) \cup (\mathbb{R} \times \{a\} \times S^2)$ .

- (a<sub>2</sub>) For all  $p \in A$  and  $k = 1, 2$ :

$$I^+(p) \cap M_k \neq \emptyset \quad \text{and} \quad I^-(p) \cap M_k \neq \emptyset$$

and

$$A \setminus (I^+(p) \cup I^-(p)) \quad \text{is compact.}$$

- (b<sub>1</sub>) We require that there exist constants  $c_{ij} > 0$  and with  $c_{00} < a$ , such that for any  $p \in M_1 \cup M_2$  and any  $\dot{g}$ -orthonormal basis  $\{e_i(p)\}_{i=0}^3 \subseteq T_p M$ , with  $e_0 = \frac{1}{\cosh x} \frac{\partial}{\partial t}$ ,

$$|h(e_i(p), e_j(p))| \leq \frac{c_{ij}}{|x(p)|}, \quad (2.1)$$

where  $h = g|_{M_1 \cup M_2} - \dot{g}|_{M_1 \cup M_2}$ .

- (b<sub>2</sub>) We further require the following decay on first derivatives of  $h$ , i.e., we assume there exists  $C_1 > 0$  such that

$$|e_k(h(e_i, e_j))(p)| \leq \frac{C_1}{|x(p)|}, \quad |e_0(h(e_i, e_j))(p)| \leq \frac{C_1}{|x(p)|^2} \quad (2.2)$$

for  $k = 1, 2, 3$  (note the faster decay on the time derivative). Additionally, we require the following decay on second derivatives,

$$|e_l(e_m(h(e_i, e_j)))(p)| \leq \frac{C_1}{|x(p)|} \quad (2.3)$$

for  $l, m = 0, \dots, 3$ .

It will be convenient to require,

$$\frac{16 \max\{c_{ij}\}}{a} < 1. \quad (2.4)$$

*Remark 2.5.* In many of the arguments involving the asymptotics (2.1)-(2.3) we will not use  $\hat{g}$ -orthonormal frames but rather work in specific charts which we will now introduce. Let  $(U, \psi)$  denote either of two charts covering  $S^2$ , with  $\psi(U) = \{(\theta, \phi) : \frac{\pi}{6} < \theta < \frac{5\pi}{6}, 0 < \phi < 2\pi\}$ , and let  $\psi : p \rightarrow (t(p), x(p), \psi(p))$  be the corresponding chart on  $M_1 \cup M_2$ . From (2.1) we see that there exists a constant  $C > 0$  such that in these charts

$$|h_{ij}| \leq \frac{C}{|x|}, |h_{ti}| \leq \frac{C \cosh(x)}{|x|}, |h_{tt}| \leq \frac{C \cosh^2(x)}{|x|} \quad (2.5)$$

for  $i, j \neq t$ . And for  $l, m, i, j$  arbitrary and  $k \neq t$

$$|\partial_k h_{ij}| \leq \frac{C \cosh^{\#t}(x)}{|x|}, |\partial_t h_{ij}| \leq \frac{C \cosh^{\#t}(x)}{|x|^2}, |\partial_l \partial_m h_{ij}| \leq \frac{C \cosh^{\#t}(x)}{|x|} \quad (2.6)$$

where  $\#t$  denotes the number of  $t$ 's appearing as lower indices.

We are also going to need some estimates for the Christoffel symbols and the curvature of  $g$ . Let now  $\#t$  denote the number of  $t$ 's appearing as lower indices minus the number of  $t$ 's appearing as upper indices. Using (2.5) and (2.6) one can show that there exists a constant  $C$  such that

$$|\hat{g}^{ij} - g^{ij}| \leq \frac{C \cosh^{\#t}(x)}{|x|}, |\hat{\Gamma}_{ij}^k - \Gamma_{ij}^k| \leq \frac{C \cosh^{\#t}(x)}{|x|} \quad (2.7)$$

and

$$|\hat{R}_{iklm} - R_{iklm}| \leq \frac{C \cosh^{\#t}(x)}{|x|}, |\hat{\text{Ric}}_{ij} - \text{Ric}_{ij}| \leq \frac{C \cosh^{\#t}(x)}{|x|}, |\hat{R} - R| \leq \frac{C}{|x|}. \quad (2.8)$$

These estimates follow in a straightforward way from (2.5) and (2.6), nevertheless their derivation is carried out in some detail in the appendix.

To simplify the constants appearing in later arguments we will always choose  $C$  such that additionally

$$C \geq \max\{c_{ij}, C_1\}. \quad (2.9)$$

## 2.4 Consequences of the asymptotics

We will start by introducing some notations: First, for any  $x_0 \in \mathbb{R}$  with  $|x_0| \geq a$  we will use the shorthand  $\{x = x_0\}$  for the submanifold  $\mathbb{R} \times \{x_0\} \times S^2$  of  $M_k$  (where  $k = 1$  for  $x_0 < 0$  and  $k = 2$  for  $x_0 > 0$ ). Further, for  $r \in \mathbb{R}_+$  and  $k = 1, 2$  we use  $M_k(r) := M_k \cap (\mathbb{R} \times \{x : |x| \geq r\} \times S^2)$  to denote the part of  $M_k$  that lies between  $\{|x| = r\}$  and infinity. We also set  $M(r) := M_1(r) \cup M_2(r)$ .

**Lemma 2.6.** *For any  $r \in [a, \infty)$  there exists  $\beta_r > 1$  and  $\alpha_r < 1$  such that on  $M(r)$*

$$\hat{g}_{\alpha_r} \prec g \prec \hat{g}_{\beta_r}$$

and one can choose  $\beta_r$  and  $\alpha_r$  such that  $\beta_r$  is decreasing in  $r$  and  $\alpha_r$  is increasing in  $r$  and  $\beta_r, \alpha_r \rightarrow 1$  as  $r \rightarrow \infty$ .

*Proof.* We first show the existence of a suitable  $\alpha_r$ . Let  $\{e_i\}_{i=0}^3$  be a  $\hat{g}$ -orthonormal basis for  $T_p M(r)$  and let  $v = v^i e_i$  be such that  $\hat{g}_\alpha(v, v) \leq 0$ . We may w.l.o.g. assume  $\sum_{i=0}^3 |v^i|^2 = 1$ . Now,  $\hat{g}_\alpha(v, v) \leq 0$  gives

$$\alpha |v^0|^2 \geq \sum_{i=1}^3 |v^i|^2 = 1 - |v^0|^2. \quad (2.10)$$

Then

$$\begin{aligned} g(v, v) &= \hat{g}(v, v) + h(v, v) = -|v^0|^2 + \sum_{i=1}^3 |v^i|^2 + h(v, v) \leq \\ &\leq (\alpha - 1)|v^0|^2 + h(v, v) \leq (\alpha - 1)|v^0|^2 + \frac{16C}{|x(p)|} \leq (\alpha - 1)|v^0|^2 + \frac{16C}{r}. \end{aligned} \quad (2.11)$$

Now if  $\alpha < 1$  we can use  $|v^0|^2 \geq \frac{1}{\alpha+1}$  to further estimate

$$g(v, v) \leq \frac{\alpha - 1}{1 + \alpha} + \frac{16C}{r}.$$

Thus, setting  $\alpha_r < \frac{1 - \frac{16C}{r}}{1 + \frac{16C}{r}} < 1$  guarantees  $g(v, v) < 0$  and since  $\frac{1 - \frac{16C}{r}}{1 + \frac{16C}{r}} \rightarrow 0$  as  $r \rightarrow \infty$  and is strictly decreasing we can choose  $\alpha_r$  to be increasing and  $\alpha_r \rightarrow 1$ .

For  $\beta_r$  we note that it suffices to show that  $\hat{g}_{\beta_r}(v, v) \geq 0$  implies  $g(v, v) > 0$ . Now  $\hat{g}_{\beta_r}(v, v) \leq 0$  gives  $1 - \sum_{i=1}^3 |v^i|^2 = |v^0|^2 \leq \frac{1}{\beta} \sum_{i=1}^3 |v^i|^2$ . So we have

$$\begin{aligned} g(v, v) &= \hat{g}(v, v) + h(v, v) \geq -|v^0|^2 + \sum_{i=1}^3 |v^i|^2 - |h(v, v)| \geq \\ &\geq \left(1 - \frac{1}{\beta}\right) \sum_{i=1}^3 |v^i|^2 - \frac{16C}{r} \geq \frac{1 - \frac{1}{\beta}}{1 + \frac{1}{\beta}} - \frac{16C}{r}. \end{aligned} \quad (2.12)$$

This implies the existence of a suitable  $\beta_r$ .  $\square$

This allows us to bound the time it takes for the entire  $S^2$ -factor to be contained in the future of a point depending on how far out (in the  $x$ -direction) this point lies.

**Lemma 2.7.** *For any  $r \in [a, \infty)$  there exists a time  $\tau_r$  such that for any  $p \in M_k(r)$*

$$\{t \geq t(p) + \tau_r\} \times \{x(p)\} \times S^2 \subseteq I^+(p),$$

$\tau_r$  is decreasing in  $r$  and  $\tau_r \rightarrow 0$  as  $r \rightarrow \infty$ .

*Proof.* Let  $\bar{\gamma} : I \rightarrow S^2$  be a unit speed geodesic (in  $S^2$ ) starting at  $\pi_{S^2}(p)$  and let  $\alpha_r$  be the constant from the previous lemma. Since  $|x(p)| \geq r$  the curve  $\gamma(s) := (t(p) + \frac{1}{\sqrt{\alpha_r \cosh(r)}}s, x(p), \bar{\gamma}(s))$  is causal for  $\mathring{g}_{\alpha_r}$ , hence timelike for  $g$  by Lemma 2.6. Noting that  $S^2$  has a finite diameter of  $\pi$  proves the claim for  $\tau_r := \frac{\pi}{\sqrt{\alpha_r \cosh(r)}}$ . This is decreasing and goes to zero as  $r \rightarrow 0$  because  $\alpha_r$  is increasing and  $\alpha_r \rightarrow 1$ .  $\square$

We also note the following consequence for null vectors.

**Lemma 2.8.** *Let  $q_{ij}$ ,  $i, j = 0, \dots, 3$ , be smooth functions on  $U \subseteq M_1 \cup M_2$  satisfying the asymptotics  $|q_{ij}| \leq \frac{C \cosh^{\#t}(x)}{|x|}$  (e.g.,  $q_{ij} = h_{ij}, \mathring{\Gamma}_{ij}^x - \Gamma_{ij}^x, \mathring{\text{Ric}}_{ij} - \text{Ric}_{ij}, \dots$ ). Then there exists a constant  $c > 0$  such that for any null vector  $v \in TU$  we have*

$$|q_{ij}v^i v^j| \leq \frac{c}{|x|}(|v^x|^2 + |\bar{v}|_{S^2}^2), \quad (2.13)$$

where  $v^i$  denotes the components of  $v$  in one of the charts  $\psi$  specified in Remark 2.5.

*Proof.* Let  $\beta_a$  be as in Lemma 2.6. Then  $v$  being null implies  $\mathring{g}_{\alpha_a}(v, v) > 0$ , which gives the estimate  $|v^t|^2 < \frac{1}{\alpha_a \cosh^2(x)}(|v^x|^2 + |\bar{v}|_{S^2}^2)$ . Further, note that in either chart  $\psi$  on  $S^2$  one always has  $|v^\theta| \leq |\bar{v}|_{S^2}$  and  $|v^\phi| < 2|\bar{v}|_{S^2}$ , which gives the estimates,

$$|v^i||v^j| \leq \frac{1}{2}(|v^i|^2 + |v^j|^2) \leq 4(|v^x|^2 + |\bar{v}|_{S^2}^2), \quad (2.14)$$

$$|v^t||v^j| \leq \frac{1}{\sqrt{\alpha_a} \cosh(x)} \sqrt{|v^x|^2 + |\bar{v}|_{S^2}^2} |v^j| \leq \frac{5}{2} \frac{1}{\sqrt{\alpha_a} \cosh(x)} (|v^x|^2 + |\bar{v}|_{S^2}^2) \quad (2.15)$$

for  $i, j \neq t$ . Hence

$$|q_{ij}v^i v^j| \leq \sum_{i,j \neq t} \frac{C}{|x|} |v^i||v^j| + 2 \sum_{j \neq t} \frac{C \cosh(x)}{|x|} |v^t||v^j| + \frac{C \cosh^2(x)}{|x|} |v^t|^2 \leq \frac{c}{|x|} (|v^x|^2 + |\bar{v}|_{S^2}^2) \quad (2.16)$$

for  $c = (36 + \frac{15}{\sqrt{\alpha_a}} + \frac{1}{\alpha_a})C$ .  $\square$

Finally we want to study maximizing null curves. Generally we say that a null curve  $\gamma : I \rightarrow M$  is a *future (or past) null ray* if  $I = [a, b)$  and  $\gamma$  is maximizing (i.e., its image is achronal) and future (resp. past) directed and future (resp. past) inextendible. We say that a null curve  $\gamma : I \rightarrow M$  is a *null line* if  $I = (a, b)$  and  $\gamma$  is maximizing and inextendible in both directions.

**Lemma 2.9** (Null rays must run to infinity). *Let  $\gamma : I \rightarrow M$  be a future null ray. Then  $\gamma$  is eventually contained in one of the  $M_k$ 's and  $|x(\gamma(s))| \rightarrow \infty$  as  $s \rightarrow b$ .*

*Proof.* If  $q \in \gamma \cap A$ , then  $\gamma$  must eventually leave the compact set  $A \setminus (I^+(q) \cup I^-(q))$  and never return to it, but since  $\gamma$  is achronal, that means that  $\gamma$  cannot return to  $A$  at all, i.e., it is contained in  $M_k$ , say  $M_2$ . So we may assume  $\gamma(0) = (t_0, x_0, \omega_0) \in M_2$  and  $x_0 = a$ . For any  $r > 0$  the set  $[t_0, t_0 + \frac{\pi}{\sqrt{\alpha_a}} + \tau_a] \times [a, r] \times S^2$  is compact, so  $\gamma$



must leave it. Since  $\gamma$  is future directed, we must have  $t > t_0$  along  $\gamma$ . Moreover, by applying, first, Lemma 2.3, then, Lemma 2.7, together with Lemma 2.6, we see that we must have  $t < t_0 + \frac{\pi}{\sqrt{\alpha_a}} + \tau_a$  along  $\gamma$ , otherwise the achronality of  $\gamma$  would be violated. It follows that  $\gamma$  must cross  $x = r$ .  $\square$

**Lemma 2.10.** *There exists  $r > 0$  such that for any null geodesic  $\gamma \subseteq M_1(r)$  one has  $\ddot{\gamma}_x > 0$ , i.e.,  $\dot{\gamma}_x$  can change sign at most once.*

*Proof.* By the geodesic equation and the estimate (2.13) we have

$$\ddot{\gamma}^x = -\Gamma_{ij}^x \dot{\gamma}^i \dot{\gamma}^j \geq -\overset{\circ}{\Gamma}_{ij}^x \dot{\gamma}^i \dot{\gamma}^j - |(\overset{\circ}{\Gamma}_{ij}^x - \Gamma_{ij}^x) \dot{\gamma}^i \dot{\gamma}^j| \geq \quad (2.17)$$

$$\geq \cosh(|x|) \sinh(|x|) |v^t|^2 - \frac{c}{|x|} (|\dot{\gamma}_x|^2 + |\dot{\gamma}|_{S^2}^2) \quad (2.18)$$

Finally,  $\gamma$  being null implies  $\hat{g}_{\beta_r}(\dot{\gamma}, \dot{\gamma}) < 0$ , which gives  $|\dot{\gamma}_t|^2 > \frac{1}{\beta_r \cosh^2(x(\gamma))} (|\dot{\gamma}_x|^2 + |\dot{\gamma}|_{S^2}^2)$ , so

$$\ddot{\gamma}^x > \left( \frac{1}{\beta_r} \tanh(|x|) - \frac{c}{|x|} \right) (|\dot{\gamma}_x|^2 + |\dot{\gamma}|_{S^2}^2) > \left( \frac{1}{\beta_r} \tanh(r) - \frac{c}{r} \right) (|\dot{\gamma}_x|^2 + |\dot{\gamma}|_{S^2}^2) > 0 \quad (2.19)$$

for  $r$  large.  $\square$

**Corollary 2.11.** *Let  $r > 0$  be such that the previous Lemma holds. Then any null geodesic  $\gamma : [a, b) \rightarrow M_1(r)$  with  $\lim_{s \rightarrow b} x(\gamma(s)) = -\infty$  may be parametrized with respect to the  $x$ -coordinate.*

*Proof.* Let  $\gamma : [a, b) \rightarrow M$  be a null geodesic with image in  $M_1(r)$ . Lemma 2.10 shows that  $\dot{\gamma}^x$  is strictly increasing, so if  $\dot{\gamma}^x(s_0) \geq 0$  for any  $s_0 \in [a, b)$  then  $\dot{\gamma}^x|_{[s_0, b)} \geq 0$  and hence  $x(\gamma|_{[s_0, b)}) \geq x(\gamma(s_0))$ . This contradicts  $\lim_{s \rightarrow b} x(\gamma(s)) = -\infty$ . Thus  $s \mapsto x(\gamma(s))$  is strictly monotonically decreasing and so there exists a reparametrization  $\tilde{\gamma} : (-\infty, x(\gamma(a))] \rightarrow M_1(r)$  of  $\gamma$  with  $x(\tilde{\gamma}(s)) = s$ .  $\square$

**Lemma 2.12.** *Any future (or past) null ray  $\gamma : [0, a) \rightarrow M$  is future (or past) complete.*

*Proof.* By the proof of Lemma 2.9 for any  $r > 0$   $\gamma$  is eventually contained in either  $M_1(r)$  or  $M_2(r)$  and  $|x(\gamma(s))| \rightarrow \infty$  as  $s \rightarrow a$ . For now, look at the case where  $\gamma$  is eventually contained in  $M_1(r)$  (for some large  $r$ , at least  $r \geq r(1)$  from the previous Lemma). We may assume  $\gamma : [0, a) \rightarrow M_1(r)$  and  $|\dot{\gamma}^x(0)| = 1$  and we have to show that  $a = \infty$ . By the arguments in Corollary 2.11 we have  $\dot{\gamma}^x(0) < 0$  and  $s \mapsto \dot{\gamma}^x(s)$  is strictly increasing, so  $|\dot{\gamma}^x(s)|^2 \leq 1$  for all  $s$ . But this gives  $|x(\gamma(s)) - x(\gamma(0))| \leq |s|$ , contradicting  $x(\gamma(s)) \rightarrow -\infty$  as  $s \rightarrow a$  if  $a < \infty$ . The case of the end contained in  $M_2(r)$  is analogous (note that the analogues to Lemma 2.10 and Corollary 2.11 show  $\ddot{\gamma}^x < 0$  and  $\dot{\gamma}^x > 0$  on  $M_2(r)$ ).  $\square$

**Lemma 2.13** (Angular velocities go to zero for null lines). *Assume the null energy condition holds, i.e.,  $\text{Ric}(X, X) \geq 0$  for all null vectors  $X$ . For any  $\varepsilon > 0$  there exists  $r(\varepsilon)$  such that  $|\dot{\gamma}|_{S^2} < \varepsilon$  on  $M_1(r(\varepsilon))$  for any null line  $\gamma : I \rightarrow M$  with  $|\dot{\gamma}^x| \leq 1$  on  $M_1(r(\varepsilon))$ .*

*Proof.* Since  $\gamma$  is complete by Lemma 2.12 the Raychaudhuri equation applied to  $\gamma$  (affinely parametrized) implies that  $\text{Ric}(\dot{\gamma}, \dot{\gamma}) = 0$  along  $\gamma$  (else  $\gamma$  would contain a pair of conjugate points). This condition does not change under reparametrization of  $\gamma$ . We now use  $\text{Ric}(\dot{\gamma}, \dot{\gamma}) = -\dot{g}(\dot{\gamma}, \dot{\gamma}) + 2|\dot{\gamma}|_{S^2}^2$  and (2.13) to estimate

$$\begin{aligned} 0 = \text{Ric}(\dot{\gamma}, \dot{\gamma}) &\geq \mathring{\text{Ric}}(\dot{\gamma}, \dot{\gamma}) - \frac{c}{|x(\gamma(s))|} (|\dot{\gamma}^x|^2 + |\dot{\gamma}|_{S^2}^2) \geq \\ &\geq -\dot{g}(\dot{\gamma}, \dot{\gamma}) + 2|\dot{\gamma}|_{S^2}^2 - \frac{c}{|x(\gamma(s))|} (1 + |\dot{\gamma}|_{S^2}^2) \geq -g(\dot{\gamma}, \dot{\gamma}) + 2|\dot{\gamma}|_{S^2}^2 - \frac{2c}{|x(\gamma(s))|} (1 + |\dot{\gamma}|_{S^2}^2) = \\ &= 2|\dot{\gamma}|_{S^2}^2 - \frac{2c}{|x(\gamma(s))|} (1 + |\dot{\gamma}|_{S^2}^2). \end{aligned} \quad (2.20)$$

So

$$\left(2 - \frac{c}{|x(\gamma(s))|}\right) |\dot{\gamma}|_{S^2}^2 \leq \frac{c}{|x(\gamma(s))|} \quad (2.21)$$

from which the claim follows.  $\square$

### 3 Proof of the main results

Throughout this section we will frequently make use of the null energy condition,  $\text{Ric}(X, X) \geq 0$  for all null vectors  $X$ . This assumption enters in Proposition 3.5 (and thus Remark 3.6) via Lemma 2.13 and in Theorem 3.9 via both Remark 3.6 and [4, Theorem IV.1]. All further results, in particular all of subsection 3.2, build upon Theorem 3.9.

#### 3.1 Constructing a foliation by totally geodesic null hypersurfaces

**Lemma 3.1.** *Let  $p \in M_k$  and  $x_0 \in (-\infty, -a] \cup [a, \infty)$ . Then  $I^\pm(p) \cap \{x = x_0\} \neq \emptyset$ .*

*Proof.* Let w.l.o.g.  $p \in M_1$  and first consider  $x_0 \in (-\infty, -a]$ . Then this is clearly true for any  $\dot{g}_\alpha$ , hence by Lemma 2.6 also for  $g$ . Since  $\{x = -a\} \subseteq \partial A \subseteq A$ , condition (a<sub>2</sub>) from Def. 2.4 then shows that  $I^\pm(p) \cap M_2 \neq \emptyset$ . Note that this also must even imply  $I^\pm(p) \cap \{x = a\} \neq \emptyset$  from which the claim follows for  $x_0 \in [a, \infty)$  by the same argument as above.  $\square$

**Lemma 3.2.** *For any  $p \in M_1$  there exists a future null ray  $\gamma_p : [0, b) \rightarrow M$  such that  $\gamma_p$  is eventually contained in  $M_2$ .*

*Proof.* By Lemma 3.1 and causality, for each positive integer  $n \in [a, \infty)$ ,  $\{x = n\} \neq I^+(p) \cap \{x = n\} \neq \emptyset$ , so there exist  $q_n \in \partial J^+(p) \cap \{x = n\}$ . Every  $q_n$  is the future endpoint of a maximizing null geodesic  $\gamma_n \subseteq \partial J^+(p)$  which must end in  $p$  because  $J^+(p)$  is closed. Hence, there exists a limit curve  $\gamma$  starting at  $p$  that is maximizing and inextendible (because the  $q_n$  run off to infinity). It is eventually contained in  $M_2$  because by the ‘no turning back’ lemma (Lemma 2.10)  $\gamma_n \cap \{x = -r\} = \emptyset$  for  $r$  large, so  $|x(\gamma(s))| \rightarrow \infty$  as  $s \rightarrow \infty$  (see Lemma 2.9) implies  $x(\gamma(s)) \rightarrow \infty$ .  $\square$

This allows us to construct null lines:

**Proposition 3.3.** *For any  $u \in \mathbb{R}$  there exists a complete null line  $\eta_u : (-\infty, \infty) \rightarrow M$  with past end eventually contained in  $M_1$  and future end eventually contained in  $M_2$  and  $t(\eta_u(s)) \rightarrow u$  as  $s \rightarrow -\infty$ .*

*Proof.* Let  $u \in \mathbb{R}$ , fix any  $\omega_0 \in S^2$  and set  $p_n := (u, -n, \omega_0) \in M_1$ . Then by Lemma 3.2 there exist maximizing future inextendible null curves  $\gamma_n : [0, \infty) \rightarrow M$  starting at  $p_n$  that are eventually contained in  $M_2$ . We now show that the sequence  $\gamma_n$  contains an accumulation point. Let  $t_{n,m}$  be the maximal  $t$ -coordinate of the set  $\gamma_n \cap \{x = -m\} \neq \emptyset$  for  $n \geq m$ . Clearly  $t_{n,m} \geq u$  for  $n \geq m$ . By Lemmas 2.6, 2.1 (and remark 2.2) and 2.7 we see that  $t_{n,m} \leq u + \tau_m + \frac{\pi}{\sqrt{\alpha_m}} - \frac{2 \tan^{-1}(e^{-n})}{\sqrt{\alpha_m}} < u + \tau_m + \frac{\pi}{\sqrt{\alpha_m}} < c$  (because all points  $p$  in  $\{x = -m\}$  with larger  $t$ -coordinate belong to  $I^+(p_n)$ ). Thus the sequence  $\{t_{n,m}\}_{n \geq m}$  has an accumulation point for  $m$  large.

By the no turning back lemma, for large enough  $m$  each  $\gamma_n$  meets  $\{x = -m\}$  in a unique point. We reparametrize such that this point is always  $\gamma_n(0)$ .

Thus there exists a limit curve  $\eta_u$  which is maximizing and both past and future inextendible, hence complete by Lemma 2.12. Since  $\gamma_n|_{[0, \infty)} \subseteq \{x \geq -m\} \cup A \cup M_2$  the same holds for  $\gamma|_{[0, \infty)}$ , so the future end of  $\gamma$  is eventually contained in  $M_2$  (by a similar argument to Lemma 3.2). And since  $\gamma_n|_{[a_n, 0]} \subseteq \{x \leq -m\} \subseteq M_1$  the past end of  $\gamma$  must lie in  $M_1$ .

Finally, we need to argue that  $t(\eta_u(s)) \rightarrow u$  as  $s \rightarrow -\infty$ . Since  $t(\gamma_n) \geq u$  (as long as  $\gamma_n$  remains in  $M_1$ ) the same holds for  $t(\eta_u)$ . Assume now that  $t(\eta_u(s)) \rightarrow u_1$  with  $u_1 > u$ . This implies that  $t(\eta_u(s)) > u_1 > u + \varepsilon$  for all  $s$ . But this is a contradiction to  $t_{n,m} \leq f_{\alpha_m}(n - m, u, -n) + \tau_m = \tau_m + \frac{\pi}{\sqrt{\alpha_m}}(\tan^{-1}(e^{-m}) - \tan^{-1}(e^{-n})) + u < u + \varepsilon$  for  $m$  large and  $n \geq m$ .  $\square$

While the construction above depends on the choice of  $\omega_0 \in S^2$  and hence is not unique, we are now going to argue that any null line  $\eta$  with  $t(\eta(s)) \rightarrow u$  is contained in a totally geodesic null hypersurface  $N_u$  that *only* depends on  $u$ . We first note the following:

**Proposition 3.4.** *Given two past inextendible causal curves  $\eta_1, \eta_2 : (-\infty, 0] \rightarrow M$  with past end contained in  $M_1$ ,  $\lim_{s \rightarrow -\infty} x(\eta_i(s)) = -\infty$  and  $\lim_{s \rightarrow -\infty} t(\eta_1(s)) > \lim_{s \rightarrow -\infty} t(\eta_2(s))$  one has  $\eta_1 \subseteq I^+(\eta_2)$ .*

*Proof.* Use Lemma 2.7, note that  $\tau_r \rightarrow 0$  as  $r \rightarrow \infty$  and that by assumption  $|x(\eta_i(s))| \rightarrow \infty$  as  $s \rightarrow -\infty$ .  $\square$

**Proposition 3.5.** *Let  $\eta_1 : (-\infty, \infty) \rightarrow M$  be a null line and  $\eta_2 : (-\infty, b] \rightarrow M$  be a past null ray, both with past end contained in  $M_1$ , such that  $\lim_{s \rightarrow -\infty} t(\eta_1(s)) = \lim_{s \rightarrow -\infty} t(\eta_2(s))$ . Then  $\eta_1 \subseteq \overline{I^+(\eta_2)}$ . If  $\eta_2$  extends to a null line this further implies  $\eta_1 \subseteq \partial J^+(\eta_2)$  and vice versa.*

*Proof.* We may assume that, far enough out, both curves are parametrized with respect to the  $x$ -coordinate, so  $|\dot{\eta}_{1,2}^x| = 1$ . We also note that by Lemma 2.13 since  $\eta_1$  is assumed to be maximizing and both future and past inextendible we have that  $|\dot{\eta}_1(s)|_{S^2}^2 \leq 1$ .

We will first show that  $\eta_1 \subseteq \overline{I^+(\eta_2)}$ . This follows immediately if we can find  $r > 0$  such that  $\eta_1|_{(-\infty, -r]}$  can be approximated by curves  ${}^\delta\eta_1 \subseteq I^+(\eta_2)$ . We are now going to construct such approximating curves.

To do this we estimate  $|g_{(t_1, x, \theta, \phi)} - g_{(t_2, x, \theta, \phi)}|$  in terms of  $|t_1 - t_2|$ . Since  $\dot{g}$  is independent of  $t$  this is just the difference of the corresponding  $h$ -terms and  $\frac{1}{\cosh(x)}|\partial_t h_{ij}| \leq \frac{C \cosh^{\#t(i,j)}}{|x|^2}$ , (2.13) gives (assuming  $v$  is null and satisfies the same estimates as  $\dot{\eta}_1$ )

$$|h_{(t_1, x, \theta, \phi)}(v, v) - h_{(t_2, x, \theta, \phi)}(v, v)| \leq |\partial_t h_{ij}| |v^i| |v^j| |t_1 - t_2| \leq \frac{2c \cosh(x)}{x^2} |t_1 - t_2|. \quad (3.1)$$

For a function  $f > 0, \dot{f} > 0$  (which will be determined later), we define the curve  ${}^\delta\eta_1 := (\eta_1^t(s) + \delta + \delta f(s), s, \bar{\eta}_1(s))$ . Clearly these curves approximate  $\eta_1$ . And by the above and  $\eta_1$  being null we may estimate  $g_{s\eta_1(s)}(\dot{\eta}_1(s), \dot{\eta}_1(s)) \leq \frac{2c \cosh(s)}{s^2}(\delta + \delta f(s))$ . This, together with  $\frac{1}{\beta_s \cosh^2(s)} \leq |\dot{\eta}_1^t(s)|^2 \leq \frac{2}{\alpha_s \cosh^2(s)}$ , leads to

$$\begin{aligned} g_{s\eta_1}({}^\delta\dot{\eta}_1, {}^\delta\dot{\eta}_1) &\leq \frac{2c \cosh(s)}{s^2}(\delta + \delta f(s)) + 2\delta g_{s\eta_1}(\dot{\eta}_1, (\dot{f}, 0, \bar{0})) + \delta^2 \dot{f}^2 g_{tt} \leq \\ &\leq \frac{2c \cosh(s)}{s^2}(\delta + \delta f(s)) + 2\delta \dot{g}(\dot{\eta}_1, (\dot{f}, 0, \bar{0})) + |2\delta \dot{f} h_{it} \dot{\eta}_1^i| \leq \\ &\leq \frac{2c \cosh(s)}{s^2}(\delta + \delta f(s)) - \frac{2\delta \cosh^2(s)}{\sqrt{\beta_s} \cosh(s)} \dot{f}(s) + |2\delta \dot{f} h_{it} \dot{\eta}_1^i|. \end{aligned} \quad (3.2)$$

Finally,  $|h_{it} \dot{\eta}_1^i| \leq |h_{tt}| |\dot{\eta}_1^t| + \sum_{i \neq t} |h_{it}| \leq (\frac{\sqrt{2}}{\sqrt{\alpha_s}} + 3) \frac{C \cosh(s)}{|s|} \leq \frac{5C \cosh(s)}{|s|}$  for  $s$  large enough to ensure  $\alpha_s > \frac{1}{2}$ . So

$$g_{s\eta_1}({}^\delta\dot{\eta}_1, {}^\delta\dot{\eta}_1) \leq \delta \cosh(s) \left( \frac{2c}{s^2} (1 + f(s)) - \frac{2}{\sqrt{\beta_s}} \dot{f}(s) + \frac{5C}{|s|} \dot{f}(s) \right) \quad (3.3)$$

Now if  $f(s) = |s|^{-\kappa}$  with  $0 < \kappa < 1$  we have that  $f$  is bounded by one and  $\dot{f}(s) \rightarrow 0$  slower than  $\frac{1}{s^2}$  as  $s \rightarrow -\infty$ . So there exists  $r$  (independent of  $\delta$ ) such that  ${}^\delta\eta_1|_{(-\infty, -r]}$

is timelike. Since by construction  $\lim t(\delta\eta_1(s)) = \lim t(\eta_1(s)) + \delta = \lim t(\eta_2(s)) + \delta$  Proposition 3.4 implies  $\delta\eta_1 \subseteq I^+(\eta_2)$ .

If both curves are null lines, we may apply the same argument to  $\eta_2$  to get  $\eta_1 \subseteq \overline{I^+(\eta_2)}$  and  $\eta_2 \subseteq \overline{I^+(\eta_1)}$ . From this we see that  $I^+(\eta_2) \subseteq I^+(\overline{I^+(\eta_1)}) = I^+(\eta_1)$ , hence  $\eta_1 \cap I^+(\eta_2) = \emptyset$  by achronality of  $\eta_1$ , proving the claim.  $\square$

*Remark 3.6.* Note that this implies that for any future directed null line  $\eta$  with past end contained in  $M_1$  the set  $\partial J^+(\eta)$  depends only on  $\lim_{s \rightarrow -\infty} t(\eta(s))$ . In particular for two lines  $\eta_{u,\omega_1}$  and  $\eta_{u,\omega_2}$  constructed as in Proposition 3.3 one has  $\partial J^+(\eta_{u,\omega_1}) = \partial J^+(\eta_{u,\omega_2}) =: \partial J^+(\eta_u)$ .

**Proposition 3.7.** *For any  $u \in \mathbb{R}$  and any  $x_0 \in (-\infty, -a] \cup [a, \infty)$  the set  $\partial J^+(\eta_u) \cap \{x = x_0\} \subseteq \mathbb{R} \times \{x_0\} \times S^2$  is a graph over  $S^2$  with continuous graphing function  $\mathcal{T}_{u,x_0} : S^2 \rightarrow \mathbb{R}$ . In particular, it is connected.*

*Proof.* Let  $\pi : \{x = x_0\} \equiv \mathbb{R} \times \{x_0\} \times S^2 \rightarrow S^2$  be the projection onto  $S^2$  and define  $S := \partial J^+(\eta_u) \cap \{x = x_0\}$ . Being the intersection of an achronal locally Lipschitz hypersurface with a timelike hypersurface,  $S$  is itself an achronal locally Lipschitz hypersurface in  $\{x = x_0\}$ . Clearly,  $\pi|_S$  is injective since  $S$  is achronal and  $\partial_t$  is timelike. Hence we may define  $\mathcal{F}_{u,x_0} := (\pi|_S)^{-1} : \pi(S) \subseteq S^2 \rightarrow S$ . Next we will argue that  $S$  is actually compact: Let  $(t_0, x_0, \omega_0) \in S$ . Then by Lemma 2.7, any  $p \in \{x = x_0\}$  with  $t(p) < t_0 - \tau_{|x_0|}$  lies in  $I^-(S)$  and any  $p \in \{x = x_0\}$  with  $t(p) > t_0 + \tau_{|x_0|}$  lies in  $I^+(S)$ . So by achronality  $S \subseteq [t_0 - \tau_{|x_0|}, t_0 + \tau_{|x_0|}] \times \{x_0\} \times S^2$ , hence it must be compact. This implies that  $\pi_S : S \rightarrow \pi(S) \subseteq S^2$  is actually a homeomorphism onto its image. In particular  $\pi(S)$  is compact and  $\mathcal{F}_{u,x_0}$  is continuous. Since  $S$  is itself a two dimensional (topological) manifold, invariance of domain implies that  $\pi|_S : S \rightarrow S^2$  is an open map. Hence  $\pi(S) = S^2$ .

Thus,  $\mathcal{F}_{u,x_0}$  is a homeomorphism, and hence  $S$  is homeomorphic to  $S^2$ , in particular connected. The graphing function  $\mathcal{T}_{u,x_0} : S^2 \rightarrow \mathbb{R}$ , defined via:  $\mathcal{T}_{u,x_0}(\omega) = t(\mathcal{F}_{u,x_0}(\omega))$  is clearly continuous.  $\square$

**Corollary 3.8.** *For any  $u \in \mathbb{R}$  the set  $\partial J^+(\eta_u)$  has only one connected component.*

*Proof.* Any point in  $\partial J^+(\eta_u)$  lies on a past inextendible achronal null geodesic  $\gamma_p$  contained in  $\partial J^+(\eta_u)$ . By the time dual of Lemma 2.9 we know that  $\gamma_p$  eventually enters  $M_1$  or  $M_2$  and hence meets  $\{x = x_0\}$  for some  $x_0 \in (-\infty, -a] \cup [a, \infty)$ . Now since  $\eta_u$  meets every  $\{x = \text{const.}\}$  slice and  $\{x = x_0\} \cap \partial J^+(\eta_u)$  is connected,  $p$  lies in the same connected component as  $\eta_u$ . Since this is true for every  $p$ , connectedness follows.  $\square$

**Theorem 3.9.** *For any  $u \in \mathbb{R}$  there exists a smooth closed achronal totally geodesic null hypersurface  $N_u$  such that there exists a null geodesic generator  $\eta$  with  $u = \lim_{s \rightarrow -\infty} t(\eta(s))$ . Further  $\lim_{s \rightarrow -\infty} t(\eta(s))$  is independent of the choice of the null generator  $\eta$  and determines  $N_u$  uniquely. We have  $N_u = \partial J^+(\eta) = \partial J^-(\eta)$ .*

*Proof.* Let  $\eta_u$  be any of the null lines from Prop. 3.3. Note that by Lemma 2.12 the null geodesic generators of  $\partial J^+(\eta_u)$  and  $\partial J^-(\eta_u)$  are complete, so we may apply the null splitting theorem [4, Theorem IV.1] to  $\eta_u$  by [4, Remark IV.2]. This gives that the connected component of  $\partial J^+(\eta_u)$  containing  $\eta_u$  is a smooth closed achronal totally geodesic null hypersurface which by construction contains a null geodesic generator  $\eta$  with  $u = \lim_{s \rightarrow -\infty} t(\eta(s))$ . Now since  $\partial J^+(\eta_u)$  and  $\partial J^-(\eta_u)$  are connected (the same arguments as in Prop. 3.7 and Cor. 3.8 also give connectedness of  $\partial J^-(\eta_u)$ ) the null splitting theorem further shows  $\partial J^+(\eta_u) = \partial J^-(\eta_u)$ . The remaining claims follow from Prop. 3.4 and Remark 3.6. Hence,  $N_u := \partial J^+(\eta_u)$  is the null hypersurface we were looking for.  $\square$

**Theorem 3.10.** *For any  $p \in M$  there exists a unique  $u_p \in \mathbb{R}$  such that  $p \in N_{u_p}$ .*

*Proof.* That  $N_{u_1} \cap N_{u_2} = \emptyset$  for  $u_1 \neq u_2$  is clear from Theorem 3.9. We need to show that for any  $p \in M$  there exists  $u_p$  such that  $p \in N_{u_p}$ . We start by showing that there exists  $r > 0$  such that  $M(r)$  is covered by the union  $\bigcup_{u \in \mathbb{R}} N_u$ .

Let  $\sigma = \mathbb{R} \times \{x_0\} \times \{\omega_0\} \subseteq M(r)$  be any  $t$ -line in  $M(r)$ . We define a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  as follows: For each  $u \in \mathbb{R}$  there is an associated totally geodesic null hypersurface  $N_u$ . By Proposition 3.7,  $N_u$  meets  $\sigma$  in a unique point; let  $f(u)$  be the  $t$ -coordinate of that point. Using Proposition 3.4, one sees (in order to avoid an achronality violation) that  $f$  is strictly increasing.

We will now argue that  $f$  is continuous and onto. Fix an interval  $[a, b]$ , and let  $u_0 \in (a, b)$ . We have  $f(a) < f(u_0) < f(b)$ . We first show that  $f$  is continuous from the left, i.e.,  $\lim_{u \rightarrow u_0^-} f(u) = f(u_0)$ .

To each  $u < u_0$  we have an associated null hypersurface  $N_u$ , and hence an associated null geodesic generator  $\eta_u$  determined by where  $N_u$  meets  $\sigma$ . Note, for  $u_1 < u_2$ , we have  $\eta_{u_2} \subset I^+(\eta_{u_1})$ . By considering their intersection with  $\sigma$  and noting that  $f(a) < f(u) = t(\eta_u \cap \sigma) < f(b)$ , we see that as  $u \nearrow u_0$  the null lines  $\eta_u$  accumulate to a unique null line  $\eta$  passing through  $\sigma$  at a  $t$ -coordinate  $t = \sup\{f(u) : u < u_0\} = \lim_{u \rightarrow u_0^-} f(u)$ . By the null splitting theorem,  $\eta$  determines a totally geodesic null hypersurface  $N_v$  for some  $v$ . Then  $\eta = \eta_v$  is the null geodesic generator of  $N_v$  determined by where  $N_v$  meets  $\sigma$ . Clearly we must have  $v \geq u_0$ , otherwise, by Proposition 3.4,  $\eta_u$  would lie to the future of  $\eta_v$  for  $u$  sufficiently close to  $u_0$ , which would contradict  $f(u) < f(v)$ . If it were the case that  $v > u_0$  then  $\eta_v$  would be in the timelike future of  $\eta_{u_0}$ . But then,  $f(v) > f(u_0)$ , so by the convergence,  $f(u) > f(u_0)$  for  $u$  sufficiently close to  $u_0$ , contradicting monotonicity of  $f$ . Hence,  $v = u_0$ , and we conclude that  $\lim_{u \rightarrow u_0^-} f(u) = f(u_0)$ .

A similar argument shows  $\lim_{u \rightarrow u_0^+} f(u) = f(u_0)$ . Thus for any  $a < c < d < b$ ,  $f$  is continuous on  $[c, d]$ , and, since increasing, onto  $[f(c), f(d)]$ . Since  $[a, b]$  is arbitrary, this is enough to imply the claim.

Thus, we have shown that every  $t$ -line in  $M(r)$  is covered by  $\bigcup_{u \in \mathbb{R}} N_u$ , so  $M(r) \subseteq \bigcup_{u \in \mathbb{R}} N_u$ . Let now  $p \in M$  be arbitrary. By a dual argument to Lemma 3.2 there exists a past inextendible maximizing ray  $\gamma_p$  that is eventually contained in  $M_1(r) \subseteq$

$\cup_{u \in \mathbb{R}} N_u$ . Now for any  $s_0$  with  $\gamma_p|_{(-\infty, s_0]} \subseteq M_1(r)$  either  $\gamma_p|_{(-\infty, s_0]} \subseteq N_{u_p}$  for  $u_p := \lim_{s \rightarrow -\infty} t(\gamma_p(s))$ , then  $p \in N_{u_p}$  since  $N_u$  is totally geodesic and we are done. Or there exists  $u \neq u_p$  with  $\gamma_p|_{(-\infty, s_0]} \cap N_u \neq \emptyset$ . If  $u > u_p$  this contradicts achronality of  $\gamma_p$  because by Prop. 3.4  $N_u \subseteq I^+(\gamma_p)$ . If  $u < u_p$  this contradicts achronality of  $N_u$  because  $\gamma_p \subseteq I^+(N_u)$ .  $\square$

*Remark 3.11.* From this we get the following structure: For any  $u \in \mathbb{R}$  the spacetime  $M$  is the disjoint union of  $I^+(N_u)$ ,  $N_u$  and  $I^-(N_u)$ . Let  $p \in M$ , then  $p \in N_{u_p}$  for some  $u_p$ . If  $u_p = u$ , then  $p \in N_u$ . If  $u_p > u$ , then  $p \in I^+(N_u)$  by Prop. 3.4. Finally, if  $u_p < u$ , then  $I^-(N_u) \cap N_{u_p} \neq \emptyset$  (by the argument in the proof of Prop. 3.4). Thus  $N_{u_p} \subseteq I^-(N_u)$  since  $N_{u_p} \cap \partial I^-(N_u) = \emptyset$  (because  $\partial I^-(N_u) = N_u$  by Theorem 3.9 and  $N_u \cap N_{u_p} = \emptyset$ ).

**Theorem 3.12.** *The null hypersurfaces  $\{N_u : u \in \mathbb{R}\}$  form a continuous codimension one foliation of  $M$ .*

*Proof.* Let  $(t, x_1, x_2, x_3)$  be coordinates on some open set  $U$  with  $\partial_t$  timelike. We will show that  $\psi : U \rightarrow \mathbb{R}^4$  defined by  $\psi(p) := (u_p, x_1(p), x_2(p), x_3(p))$  is a continuous chart on  $U$ , for which clearly  $\{p \in U : u_p = c\} = N_c \cap U$ . Further,  $p \mapsto u_p$  is continuous on  $M$ : Let  $p_n \rightarrow p_0$ , then the null lines  $\eta_n \subseteq N_{u_{p_n}}$  corresponding to  $p_n$  accumulate to a null line  $\eta \subseteq N_{u_{p_0}}$  at  $p_0$ . From this continuity follows as in the previous proof. Finally,  $\psi$  is injective. Assume  $\psi(p_1) = \psi(p_2)$ , then  $x_i(p_1) = x_i(p_2)$  for  $i = 1, 2, 3$  and it remains to show that  $t(p_1) = t(p_2)$ . If not, w.l.o.g.  $t(p_1) > t(p_2)$  so, by  $t$  being the time coordinate,  $p_1 \in I^+(p_2)$  which contradicts  $u_{p_1} = u_{p_2}$  by achronality of the  $N_u$ 's. From this invariance of domain implies that  $\psi$  is a homeomorphism, i.e., a continuous chart.  $\square$

## 3.2 Obtaining a foliation by totally geodesic round 2-spheres

The same way one constructed the foliation  $\{N_u\}_{u \in \mathbb{R}}$  one may obtain a second, transverse foliation with the same properties except that its null geodesic generators will be past instead of future directed. We denote this transverse foliation by  $\{\hat{N}_v\}_{v \in \mathbb{R}}$ . The idea is now to show that  $S_{u,v} := N_u \cap \hat{N}_v$  (if non-empty) are isometric 2-spheres and to use the asymptotics to argue that they must even be isometric to round 2-spheres.

We will first aim to characterize the pairs  $(u, v)$  for which  $S_{u,v} \neq \emptyset$ . To do so, let  $\eta_u$  be a future directed null geodesic generator of  $N_u$  and  $\hat{\eta}_v$  be a past directed null geodesic generator for  $\hat{N}_v$ . Then we define  $u_\infty := \lim_{s \rightarrow \infty} t(\eta_u(s))$  and  $v_\infty := \lim_{s \rightarrow \infty} t(\hat{\eta}_v)$ . These do not depend on the choice of  $\eta_u, \hat{\eta}_v$  by an analogue of Prop. 3.5.

**Lemma 3.13.** *Let  $(u, v) \in \mathbb{R}^2$ . Then the following are equivalent:*

- (i)  $S_{u,v} \neq \emptyset$ ,
- (ii)  $u < v$  and  $u_\infty > v_\infty$ ,

(iii) any (future directed) null geodesic generator of  $N_u$  starts in  $I^-(\hat{N}_v)$  and ends in  $I^+(\hat{N}_v)$ ,

(iv) any null geodesic generator of  $N_u$  meets  $S_{u,v}$  exactly once.

*Proof.* We begin by showing that (i) implies (ii). If  $S_{u,v} \neq \emptyset$  then there exists a (future directed) null geodesic generator  $\eta_u$  of  $N_u$  with  $\eta_u \cap \hat{N}_v \neq \emptyset$ . Since  $\eta_u$  intersects  $\hat{N}_v$  transversally and  $M = I^+(\hat{N}_v) \cup \hat{N}_v \cup I^-(\hat{N}_v)$  (by the analogue of Remark 3.11) it intersects  $\hat{N}_v$  only once, say in  $\eta_u(s_0)$ . Because it is future directed we have  $\eta_u|_{(-\infty, s_0)} \subseteq I^-(\hat{N}_v)$  and  $\eta_u|_{(s_0, \infty)} \subseteq I^+(\hat{N}_v)$ . So there exists  $r$  large such that  $\eta_u \cap M_1(r) \subseteq I^-(\hat{N}_v)$ . Then  $t(\eta_u \cap \{x = -r\}) < t(\hat{\eta}_v \cap \{x = -r\})$  for an appropriately chosen (past directed) generator of  $\hat{N}_v$ . This gives  $u < v$  because  $t$  is decreasing along  $\eta_u$  and increasing along  $\hat{\eta}_v$  as  $s \rightarrow -\infty$ . An analogous argument in  $M_2(r)$  shows  $u_\infty > v_\infty$ .

Now, if  $u < v$  and  $u_\infty > v_\infty$  it immediately follows from (a slight variation of) Prop. 3.4 that any null geodesic generator of  $N_u$  starts in  $I^-(\hat{N}_v)$  and ends in  $I^+(\hat{N}_v)$ . This shows (iii).

If any null geodesic generator of  $N_u$  starts in  $I^-(\hat{N}_v)$  and ends in  $I^+(\hat{N}_v)$  then it must intersect  $\hat{N}_v = \partial I^+(\hat{N}_v)$  and hence  $S_{u,v}$  at least once. Further, it can intersect  $S_{u,v}$  at most once by the same argument as in the first paragraph. This shows (iv).

Finally, that (iv) implies (i) is obvious.  $\square$

**Proposition 3.14.** *For any  $(u, v) \in \mathbb{R}^2$  with  $u < v$  and  $u_\infty > v_\infty$  the set  $S_{u,v}$  is a totally geodesic, spacelike codimension 2 submanifold homeomorphic to  $S^2$ . Further for any two such pairs  $u_1, v_1$  and  $u_2, v_2$  the spheres  $S_{u_1, v_1}$  and  $S_{u_2, v_2}$  are isometric.*

*Proof.* That the intersection is a totally geodesic, (smooth) spacelike codimension 2 submanifold follows immediately from  $N_u$  and  $\hat{N}_v$  intersecting transversally and being totally geodesic.

Let  $n_u$  be a null vectorfield defining  $N_u$  with  $\nabla_{n_u} n_u = 0$ . Then its flow  $\Phi^{n_u} : \mathbb{R} \times (N_u \cap \{x = -r\}) \rightarrow N_u$  is a diffeomorphism (for  $r$  sufficiently large): By Lemma 2.9 and Corollary 2.11 every integral curve of  $n_u$  intersects  $N_u \cap \{x = -r\}$  exactly once and clearly every point of  $N_u$  lies on an integral curve. Since by Lemma 3.13 any integral curve also intersects  $S_{u,v}$  exactly once we may rescale  $n_u$  such that  $\Phi^{n_u}(1, \cdot) : \{x = -r\} \cap N_u \rightarrow S_{u,v}$  is a diffeomorphism. Thus  $S_{u,v}$  is homeomorphic to  $S^2$  by Prop. 3.7.

Next we show that  $S_{u, v_1}$  is isometric to  $S_{u, v_2}$  if both are non-empty. This follows by a fairly standard argument from the fact that  $N_u$  is totally geodesic (see e.g [1, Appendix A]): We rescale  $n_u$  such that  $\Phi_1^{n_u} \equiv \Phi^{n_u}(1, \cdot) : S_{u, v_1} \rightarrow S_{u, v_2}$  is a diffeomorphism. Let  $X_1, X_2$  be a basis for  $T_p S_{u, v_1}$ . We need to show that  $g(X_i, X_j) = g((\Phi_1^{n_u})_*(X_i), (\Phi_1^{n_u})_*(X_j))$ . For  $s \in (0, 1]$  we set  $X_i(s) := (\Phi_s^{n_u})_* X_i$ . Then a straightforward computation shows  $\mathcal{L}_{n_u} X_i = 0$ . Setting  $g_{ij}(s) = g(X_i(s), X_j(s))$  we



have

$$\begin{aligned} \frac{d}{ds} g_{ij} &= n_u(g(X_i, X_j)) = g(\nabla_{n_u} X_i, X_j) + g(X_i, \nabla_{n_u} X_j) = \\ &= g(\nabla_{X_i} n_u, X_j) + g(X_i, \nabla_{X_j} n_u) = 0, \end{aligned} \quad (3.4)$$

because the null second fundamental form of  $N_u$  vanishes since  $N_u$  is totally geodesic. The same argument (only using  $\hat{N}_v$  instead of  $N_u$ ) applies to show that  $S_{u_1, v}$  and  $S_{u_2, v}$  are also isometric. Since one can see that any two (non-empty) spheres  $S_{u_1, v_1}$  and  $S_{u_2, v_2}$  can be connected via finitely many steps of this form.  $\square$

Now we will estimate the curvature of such spheres  $S_{u, v} \subseteq M(r)$  for  $r$  large.

**Proposition 3.15.** *For any  $\varepsilon > 0$  there exists  $r$  (depending only on  $\varepsilon$ ) such that any (non-empty)  $S_{u, v} \subseteq M(r)$  has Gauss curvature  $1 - \varepsilon \leq K_{u, v} \leq 1 + \varepsilon$ .*

*Proof.* Let  $n_u$  and  $\hat{n}_v$  be null vector fields defining  $N_u$  and  $\hat{N}_v$ , respectively. We assume that they are normalized to  $n_u^x = \hat{n}_v^x = 1$ . To simplify notation, we will drop the indices  $u, v$ . Let  $X \in TS$  be any vector tangent to  $S$  with  $|X^x|^2 + |\bar{X}|_{S^2}^2 \leq 1$  and hence in our charts  $|X^\theta| \leq 1, |X^\phi| \leq 2$ . We will first estimate  $|X^t|$ : Since  $X \in TS$  we have  $g(X, n - \hat{n}) = 0$  and  $X$  is  $g$ -spacelike, so  $\dot{g}_{\alpha r}(X, X) > 0$ , i.e.,  $|X^t| \leq \frac{1}{\sqrt{\alpha_r \cosh(x)}} \leq \frac{2}{\cosh(x)}$  for  $r$  large. Thus, we estimate

$$\begin{aligned} |\dot{g}(X, n - \hat{n}) - g_{S^2}(\bar{X}, \bar{n} - \bar{\hat{n}})| &\leq |h(X, n - \hat{n})| + g_{S^2}(\bar{X}, \bar{n} - \bar{\hat{n}}) \leq \\ &\leq |h_{ij}| |X^i| |n^j - \hat{n}^j| + 2 \sin^2(\theta) |\bar{n}^\phi - \bar{\hat{n}}^\phi| + 2 |\bar{n}^\theta - \bar{\hat{n}}^\theta| \leq \\ &\leq \frac{6C \cosh(x)}{r} |n^t - \hat{n}^t| + \frac{6C}{r} \sum_{j=1}^3 |n^j - \hat{n}^j| + 2 \sin^2(\theta) |\bar{n}^\phi - \bar{\hat{n}}^\phi| + 2 |\bar{n}^\theta - \bar{\hat{n}}^\theta| = \\ &= \frac{6C \cosh(x)}{r} |n^t - \hat{n}^t| + \frac{6C}{r} \sum_{j \neq \theta, \phi} |n^j - \hat{n}^j| + 2 \sin^2(\theta) |\bar{n}^\phi - \bar{\hat{n}}^\phi| + 2 |\bar{n}^\theta - \bar{\hat{n}}^\theta|. \end{aligned} \quad (3.5)$$

Using Lemma 2.13 we see that for any  $\varepsilon$  there exists  $r$  such that  $|\bar{n}|_{S^2}^2, |\bar{\hat{n}}|_{S^2}^2 < \varepsilon^2$ , i.e., in our charts  $|n^\phi| < 2\varepsilon, |\hat{n}^\phi| < 2\varepsilon$  and  $|n^\theta| < \varepsilon, |\hat{n}^\theta| < \varepsilon$ . By Lemma 2.6 we estimate  $|n^t|^2, |\hat{n}^t|^2 < \frac{1+\varepsilon^2}{\alpha_r \cosh^2(x)}$ . Thus

$$|-\cosh^2(x) X^t (n^t - \hat{n}^t)| \leq \frac{6C}{r} (6\varepsilon + 2\sqrt{\frac{1+\varepsilon^2}{\alpha_r}}) + 12\varepsilon.$$

Because  $n$  is future pointing and  $\hat{n}$  is past pointing we see that  $|n^t - \hat{n}^t| = n^t + |\hat{n}^t| \geq \frac{2}{\sqrt{\beta_r \cosh(x)}}$ , so  $|X^t| \leq \frac{1}{\cosh(x)} (\frac{c}{r} + c'\varepsilon)$  for some  $c, c'$ . Thus for any  $\varepsilon > 0$  there exists  $r$  such that

$$|X^t| \leq \frac{1}{\cosh(x)} \varepsilon. \quad (3.6)$$

Next we derive a similar estimate for  $|X^x|$ : Proceeding as before but looking at  $\mathring{g}(X, n + \hat{n})$  yields

$$|-\cosh^2(x)X^t(n^t + \hat{n}^t) + 2X^x| \leq \left(\frac{c}{r} + c'\varepsilon\right) \quad (3.7)$$

for some  $c, c'$ . We now need to estimate  $|n^t + \hat{n}^t|$ . Since  $\frac{1}{\sqrt{\beta_r \cosh(x)}} \leq |n^t|, |\hat{n}^t| \leq \frac{1+\varepsilon}{\sqrt{\alpha_r \cosh(x)}}$  we have  $|n^t + \hat{n}^t| = |n^t - |\hat{n}^t|| \leq \frac{1}{\cosh(x)} \left(\frac{1+\varepsilon}{\sqrt{\alpha_r}} - \frac{1}{\sqrt{\beta_r}}\right) \leq \frac{1}{\cosh(x)}\varepsilon$  for  $r$  large. Combining this with (3.6) we see  $|\cosh^2(x)X^t(n^t + \hat{n}^t)| < \varepsilon^2$  and hence for any  $\varepsilon > 0$  we can find  $r$  such that also

$$|X^x| \leq \varepsilon. \quad (3.8)$$

Note that these estimates also ensure that  $S$  is  $\mathring{g}$ -spacelike.

We now use this to estimate  $K$ : Let  $p \in S$ , let  $X, Y$  be a  $\mathring{g}$ -orthogonal basis for  $T_p S$  with  $|X^x|^2 + |\bar{X}|_{S^2}^2 = 1$  and denote the Riemann tensor for  $(S, g|_S)$  by  $R_S$ . Then

$$K(X, Y) = \frac{g(R_S(X, Y)Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2} = \frac{g(R(X, Y)Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2} \quad (3.9)$$

because  $S$  is totally geodesic in  $M$ .

We start by estimating  $K(X, Y) - \mathring{K}(X, Y)$  (where

$$\mathring{K}(X, Y) = \frac{\mathring{g}(\mathring{R}(X, Y)Y, X)}{\mathring{g}(X, X)\mathring{g}(Y, Y)}).$$

First note that by (3.6)

$$|\mathring{g}(X, X) - 1| = |\mathring{g}(X, X) - |X^x| - |\bar{X}|_{S^2}^2| = \cosh^2(x)|X^t|^2 < \varepsilon \quad (3.10)$$

and the same for  $|\mathring{g}(Y, Y) - 1|$ , i.e.,  $X$  and  $Y$  are close to being  $\mathring{g}$ -orthonormal. From  $|\mathring{R}_{ijkl} - R_{ijkl}| \leq \frac{C \cosh^{\#t}(x)}{r}$  (see Remark 2.5) and using (3.6), (3.8) for  $X, Y$  and  $|X^i|, |Y^i| \leq 2$  for  $i = \theta, \phi$  (which follows from our choice of charts and  $\bar{X}, \bar{Y}$  having unit  $g_{S^2}$ -norm) we see that

$$|\mathring{g}(\mathring{R}(X, Y)Y, X) - g(R(X, Y)Y, X)| \leq \frac{c}{r} \quad (3.11)$$

for some  $c > 0$ . Similarly, using  $|h_{ij}| \leq \frac{C \cosh^{\#t}(x)}{r}$ , we get  $|g(X, Y) - \mathring{g}(X, Y)| < \frac{c}{r}$  and

$$1 + \frac{c}{r} + \varepsilon \leq |g(X, X)|, |g(Y, Y)| \leq 1 - \frac{c}{r} - \varepsilon \quad (3.12)$$

(note that  $\mathring{g}(X, X), \mathring{g}(Y, Y) \in (1 - \varepsilon, 1 + \varepsilon)$ ). Putting these estimates together shows that indeed for any  $\varepsilon$  there exists  $r$  such that

$$|K(X, Y) - \mathring{K}(X, Y)| \leq \varepsilon$$

as long as  $S \subseteq M_1(r) \cup M_2(r)$ .

To estimate  $\mathring{K}(X, Y)$  note that because  $\mathring{g} = g_{AdS_2} + g_{S^2}$ ,  $K_{S^2} = 1$  and (3.6),(3.8) we have

$$\begin{aligned} |\mathring{g}(\mathring{R}(\bar{X}, \bar{Y})\bar{Y}, \bar{X}) - 1| &= |\mathring{g}(\bar{X}, \bar{X})\mathring{g}(\bar{Y}, \bar{Y}) - \mathring{g}(\bar{X}, \bar{Y})^2 - 1| \leq \\ &\leq |\mathring{g}(X, X)\mathring{g}(Y, Y) - \mathring{g}(X, Y)^2 - 1| + c\varepsilon. \end{aligned} \quad (3.13)$$

So for any  $\varepsilon > 0$  we can find  $r$  such that

$$|\mathring{g}(\mathring{R}(\bar{X}, \bar{Y})\bar{Y}, \bar{X}) - 1| < \varepsilon. \quad (3.14)$$

Finally,

$$|\mathring{K}(X, Y) - 1| = |\mathring{g}(\mathring{R}(X, Y)Y, X) - 1| \leq |\mathring{g}(\mathring{R}(\bar{X}, \bar{Y})\bar{Y}, \bar{X}) - 1| + c\varepsilon < (c+1)\varepsilon \quad (3.15)$$

and we are done.  $\square$

**Theorem 3.16.** *The family  $\{S_{u,v}\}_{(u,v) \in Q}$ , where  $Q = \{(u, v) \in \mathbb{R}^2 : u < v \text{ and } u_\infty > v_\infty\}$ , gives a continuous foliation of  $M$  by totally geodesic round 2-spheres.*

*Proof.* We start by showing that  $S_{u,v}$  is isometric to  $S^2$ . Let  $r_0$  be large enough for Lemma 2.10 to apply. We will show that for any  $(u, v) \in Q$  and  $r > a$  there exists  $u_0 \equiv u_0(v, r)$  such that  $[u, u_0] \times \{v\} \subseteq Q$  and  $S_{u_0,v} \cap M_1(r_0) \subseteq M_1(r)$ . For any past null generator  $\eta_u : (-\infty, x_p] \rightarrow M$  of  $N_u$  starting in a point  $p = (t_p, x_p, \omega_p) \in M_1(r_0)$  with  $|x_p| = r$  we have  $\eta_u(s) \in J^-(t_p + \tau_r, x_p, \omega(\eta_u(s)))$  by Lemma 2.7. So, since such a generator must be contained in  $M_1(r)$ , we get  $\eta_u(s) \in I_{\mathring{g}_{\beta_r}}^-(t_p + \tau_r, x_p, \omega(\eta_u(s)))$  by Lemma 2.6, and hence by Lemma 2.1

$$t(\eta_u(s)) < \frac{2}{\sqrt{\beta_r}}(\tan^{-1}(e^s) - \tan^{-1}(e^{-r})) + t_p + \tau_r \quad (3.16)$$

if  $\eta_u$  is parametrized with respect to the  $x$ -coordinate. Letting  $s \rightarrow -\infty$  we get

$$t_p \geq u + \frac{2}{\sqrt{\beta_r}} \tan^{-1}(e^{-r}) + \tau_r. \quad (3.17)$$

A similar argument applied to  $\hat{\eta}_v : (-\infty, x_p] \rightarrow M$ , using that  $\hat{\eta}_v(s) \in J^+(t_p - \tau_r, x_p, \omega(\hat{\eta}_v(s)))$ , shows

$$t_p \leq v - \frac{2}{\sqrt{\beta_r}} \tan^{-1}(e^{-r}) - \tau_r. \quad (3.18)$$

So if  $p \in S_{u,v}$ , then

$$\frac{4}{\sqrt{\beta_r}} \tan^{-1}(e^{-r}) + 2\tau_r \leq v - u. \quad (3.19)$$

Hence by choosing  $u_0(v, r) < v$  as close to  $v$  as necessary it follows that  $S_{u_0,v} \cap M_1(r_0) \subseteq M_1(r)$ . That  $[u, u_0] \times \{v\} \subseteq Q$  is clear from  $u \mapsto u_\infty$  being increasing, so  $\bar{u}_\infty > u_\infty > v_\infty$  for all  $\bar{u} > u$ .

Now connectedness of  $S_{u_0(r),v}$  implies that even  $S_{u_0(r),v} \subseteq M_1(r)$  for any  $r > r_0$ . Then by Prop. 3.15 the Gauss curvature  $K_{u_0(r),v} \rightarrow 1$  uniformly on  $S_{u_0(r),v}$  as  $r \rightarrow \infty$ . But because all the  $S_{u_0(r),v}$  are isometric to  $S_{u,v}$ , their Gauss curvatures (in corresponding points) have to be equal, so  $K_{u,v} = 1$ . Together with  $S_{u,v}$  being homeomorphic to  $S^2$  this shows that  $S_{u,v}$  is isometric to the round 2-sphere.

It remains to show that  $\{S_{u,v}\}_{(u,v) \in Q}$  is a continuous foliation. This follows from the Frobenius theorem if we can show that  $p \mapsto T_p S_{u_p, v_p}$  is a continuous distribution (it clearly is integrable, because it consists of tangent spaces to (smooth) submanifolds). Since  $T_p S_{u_p, v_p} = \text{span}\{n_p, \hat{n}_p\}^\perp$ , where  $n_p$  and  $\hat{n}_p$  denote the future pointing null tangents (normed w.r.t. some Riemannian background metric) to  $N_{u_p}$  and  $\hat{N}_{v_p}$  in  $p$ , it is sufficient to show continuity of  $p \mapsto n_p$  and  $p \mapsto \hat{n}_p$ . Let  $p_k \rightarrow p_0$  and let  $\eta_n$  be the unique null geodesic generators of  $N_{u_{p_k}}$  with  $\dot{\eta}_k(0) = n_{p_k}$ . Then the  $\eta_k$  accumulate to a null line  $\eta_0$  passing through  $p_0$  with  $\eta \subseteq N_{u_{p_0}}$ . Hence  $n_{p_k} = \dot{\eta}_k(0) \rightarrow \dot{\eta}_0(0) = n_{p_0}$ .  $\square$

## 4 Asymptotically $AdS_2 \times S^2$ spacetimes with parallel Ricci tensor

In this section we will use the assumption of  $\nabla \text{Ric} = 0$  to first obtain a general local splitting result, see Thm. 4.1, and finally a full rigidity result, see Thm. 4.4. For  $k > 0$  we denote by  $AdS_2(k)$  and  $dS_2(k)$  two dimensional anti-de Sitter space with scalar curvature  $-2k$  and two dimensional de Sitter space with scalar curvature  $2k$ , respectively. Similarly  $S^2(k)$  and  $H^2(k)$  denote the two dimensional sphere with scalar curvature  $2k$  and two dimensional hyperbolic space with scalar curvature  $-2k$ .

**Theorem 4.1.** *Let  $(M, g)$  be a (four dimensional, connected) spacetime with  $\nabla \text{Ric} = 0$ . If  $R = 0$  and  $\text{Ric}$  is non-degenerate, then there exists  $k > 0$  such that any  $p \in M$  has a neighbourhood  $U$  that is isometric to an open subset  $V$  of  $AdS_2(k) \times S^2(k)$  or  $dS_2(k) \times H^2(k)$ .*

*Proof.* First note that  $\text{Ric}$  cannot be proportional to the metric because  $R = 0$  but  $\text{Ric} \neq 0$  because it is non-degenerate. So [13, Lemma 3.1] applies showing that for any open simply connected domain  $(D, g) \subseteq (M, g)$  either the holonomy group is non-degenerately reducible or there exists a covariantly constant null vector field  $X$ . But by the definition of  $\text{Ric}$  one clearly has  $\text{Ric}(X, Y) = 0$  for any vector field  $Y$  if  $\nabla X = 0$ . So the existence of a covariantly constant vector field contradicts the non-degeneracy of  $\text{Ric}$ . Hence the holonomy group of  $(D, g)$  is non-degenerately reducible.

Now [15, Prop. 3] gives that any point  $p$  in  $M$  has a neighbourhood  $U$  that is isometric to a direct product, say  $U = L \times P$ , where  $L$  is Lorentzian and  $P$  is Riemannian. First note that  $\text{Ric}_L$  and  $\text{Ric}_P$  are non-degenerate (as bilinear forms on  $TL \times TL$ , respectively  $TP \times TP$ ): By the direct product structure  $\text{Ric}(X, Y) = 0$  for  $X \in TL$  and  $Y \in TP$  so if  $\text{Ric}_L$  or  $\text{Ric}_P$  were degenerate, then so would be  $\text{Ric}$ . Thus,  $\dim(L) > 1$  and  $\dim(P) > 1$ , so the only possibility is  $\dim(L) = 2 = \dim(P)$ .

Neither  $L$  nor  $P$  splits, since any further splitting would give a one-dimensional factor contradicting non-degeneracy of Ric. So both  $\text{Ric}_L$  and  $\text{Ric}_P$  have to be proportional to the respective metrics on  $L$  and  $P$ , i.e.,  $\text{Ric}_L = \lambda_L g_L$  and  $\text{Ric}_P = \lambda_P g_P$  with  $\lambda_L + \lambda_P = 0$ . Setting  $k := |\lambda_P| = |\lambda_L|$ , non-degeneracy of Ric implies  $k > 0$ . So we have shown that for any  $p \in M$  there exists a  $k$  and a neighbourhood  $U$  that is isometric to an open subset  $V$  of  $AdS_2(k) \times S^2(k)$  (if  $\lambda_P > 0$ ) or  $dS_2(k) \times H^2(k)$  (if  $\lambda_P < 0$ ). Clearly  $\lambda_P$ , and thus  $k$ , is unique and locally constant, hence constant.  $\square$

*Remark 4.2.* It is actually sufficient to assume that there exists a point  $p_0$  such that  $\text{Ric}_{p_0}$  is non-degenerate and a sequence  $p_n$  such that  $R_{p_n} \rightarrow 0$ . This is obvious from the fact that  $\nabla \text{Ric} = 0$  implies  $\nabla R = 0$ , so  $R = \text{const.}$ , and that if  $\text{Ric}_{p_0}(X_{p_0}, \cdot) = 0$  then  $\text{Ric}(X, \cdot) = 0$  for any  $X$  that is the parallel transport of  $X_{p_0}$  along any curve.

If  $(M, g)$  is asymptotically  $AdS_2 \times S^2$ , then  $\lambda_P = 1$  and the structure obtained in the previous section is consistent with this local product structure.

**Corollary 4.3.** *Let  $(M, g)$  be asymptotically  $AdS_2 \times S^2$  (in the sense of Def. 2.4) and assume that the null energy condition holds and that  $\nabla \text{Ric} = 0$ . Then any  $p \in M$  has a neighbourhood  $U$  that is isometric to an open subset  $V \equiv L \times P$  of  $AdS_2 \times S^2$  (with metric  $\hat{g}$ ). Further, the tangent space  $T_q L$  is spanned by the vectors  $n_q, \hat{n}_q$  and  $T_q P = T_q S_{u_q, v_q}$  for all  $q \in V$ .*

*Proof.* Clearly  $R_{p_n} \rightarrow 0$  as  $x(p_n) \rightarrow \infty$  by the asymptotics (2.8). Also, there must exist a point  $p$  where  $\text{Ric}_p$  is non-degenerate: Else we can find a sequence  $p_n \in M_2$  with  $x(p_n) \rightarrow \infty$  and vectors  $X_n \in T_{p_n} M$  with  $\text{Ric}(X_n, \cdot) = 0$ . We may assume that these  $X_n$  are normed to  $\cosh^2(x(p_n))|X_n^t|^2 + |X_n^x|^2 + |\bar{X}_n|_{S^2}^2 = 1$ , so setting  $Y_n := X_n^t \partial_t - X_n^x \partial_x + \bar{X}_n$  we have  $\text{Ric}(X_n, Y_n) = 1$  and  $|\text{Ric}(X_n, Y_n) - \text{Ric}(X_n, Y_n)| \leq \frac{C}{|x(p_n)|}$ . This contradicts  $\text{Ric}(X_n, \cdot) = 0$  for large enough  $x(p_n)$ .

Thus, by Remark 4.2, we can apply Theorem 4.1, to get  $U \cong L \times P$ . We have that  $n, \hat{n} \subseteq TL$ : If not, then  $0 = g_L(n, n) + g_P(n, n)$  and  $g_P(n, n) \neq 0$ , so  $-g_L(n, n) = g_P(n, n) > 0$  because  $g_P$  is Riemannian. So  $\text{Ric}(n, n) = -k g_L(n, n) + k g_P(n, n) \neq 0$ , contradicting  $\text{Ric}(n, n) = 0$  (which follows from the NEC and  $n, \hat{n}$  being tangent to null lines). Thus  $T_q L$  is spanned by  $n_q, \hat{n}_q$  and  $T_q S_{u_q, v_q} = \text{span}\{n_q, \hat{n}_q\}^\perp = T_q P$ . Finally, because  $S_{u, v}$  is isometric to the round 2-sphere by Theorem 3.16, we must have  $\lambda_P = 1$ , so  $L \times P \subseteq AdS_2 \times S^2$ .  $\square$

Finally, the fact that the spheres  $S_{u, v}$  are isometric to  $S^2$  and hence geodesically complete allows us to globalize this splitting:

**Theorem 4.4.** *Let  $(M, g)$  be asymptotically  $AdS_2 \times S^2$  (in the sense of Def. 2.4) and assume that the null energy condition holds and that  $\nabla \text{Ric} = 0$ . Then  $M$  is isometric to  $AdS_2 \times S^2$ .*

*Proof.* From the local splitting in Cor. 4.3 we see that the foliation  $F := \{S_{u, v}\}_{(u, v) \in Q}$  from Theorem 3.16 must be smooth. Further, the distribution  $q \mapsto \text{span}\{n_q, \hat{n}_q\} =$

$T_q S_{u_q, v_q}^\perp \subseteq T_q M$  must be smooth as well and hence by the Frobenius theorem give rise to a smooth foliation  $K$  with leaves perpendicular to the leaves of  $F$ . Also, from the local product structure we immediately see that both of these foliations are totally geodesic (i.e., their leaves are totally geodesic).

For  $F$  we know even more: Note that the leaves are exactly the spheres  $S_{u,v}$  which are totally geodesic submanifolds isometric to  $(S^2, d\Omega^2)$  by Theorem 3.16 and hence even geodesically complete. Finally, note the  $M$  is simply connected because it is homeomorphic to  $\mathbb{R}^2 \times S^2$  (for any  $x_0 < -a$  the flow  $\Phi^n : \mathbb{R} \times (\mathbb{R} \times \{x = x_0\} \times S^2) \rightarrow M$  of  $n$  is a homeomorphism).

So we may apply [12, Cor. 2] to obtain that  $M$  is globally isometric to a product  $L \times P$  such that  $K$  and  $F$  correspond to the canonical foliations of the product  $L \times P$ . Since  $P$  is a leaf of  $F$ , we see that  $P = (S^2, d\Omega^2)$ . And since  $L$  is a leaf of  $K$  it must be isometric to a non-empty open subset  $U$  of  $(AdS_2, g_{AdS_2})$ . Further  $L$  is null geodesically complete because the only null geodesics in  $Q$  are null geodesic generators of the achronal null hypersurfaces  $N_u$  and  $\tilde{N}_v$ , hence complete by Lemma 2.12. So all that remains is to show that any null geodesically complete non-empty open subset  $U$  of  $AdS_2$  must already be all of  $AdS_2$ : For any  $p \in AdS_2 \setminus U$  all null geodesics emanating from  $p$  must also lie in  $AdS_2 \setminus U$ . So if  $U \neq AdS_2$  then  $AdS_2 \setminus U = AdS_2$  because any two points in  $AdS_2$  can be connected by a curve consisting solely of null geodesic segments.  $\square$

## A Asymptotics for the curvature

In this appendix we give some details on the derivation of (2.7), (2.8) from (2.5), (2.6). Throughout this appendix we use  $C$  to denote a running constant.

In general, if two pairs of functions  $\mathring{f}_1, f_1$  and  $\mathring{f}_2, f_2$  satisfy  $|f_1 - \mathring{f}_1| \leq \frac{C}{|x|} |\mathring{f}_1(x)|$  and  $|f_2 - \mathring{f}_2| \leq \frac{C}{|x|} |\mathring{f}_2(x)|$  on  $\mathbb{R} \setminus [-a, a]$  then  $|f_1| \leq C|\mathring{f}_1|$ ,  $|f_2| \leq C|\mathring{f}_2|$  and

$$|\mathring{f}_1 \mathring{f}_2 - f_1 f_2| \leq \frac{C}{|x|} |\mathring{f}_1(x) \mathring{f}_2(x)| \text{ on } \mathbb{R} \setminus [-a, a]. \quad (\text{A.1})$$

Using this, (2.5) and the form of  $\mathring{g}$  (note that  $\sin(\theta)$  is bounded away from zero in the charts we use) allows us to estimate

$$\begin{aligned} |\det(\mathring{g}) - \det(g)| &\leq |\det(\mathring{g}) - \prod_{i=1}^4 g_{ii}| + \sum_{\sigma \neq \text{id}} \prod_{i=1}^4 |g_{i\sigma(i)}| \leq \\ &\leq \frac{C}{|x|} \cosh^2(x) + \frac{C \cosh^2(x)}{|x|^2} \left(1 + \frac{1}{|x|} + \frac{1}{|x|^2}\right) \leq \frac{C}{|x|} \cosh^2(x). \end{aligned} \quad (\text{A.2})$$

From this we get

$$\left| \frac{1}{\det(\mathring{g})} - \frac{1}{\det(g)} \right| \leq \frac{C}{|x|} \frac{1}{\cosh^2(x)} \quad (\text{A.3})$$

and using  $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$  this gives

$$|\mathring{g}^{tt} - g^{tt}| \leq \frac{C}{|x|} \frac{1}{\cosh^2(x)}, \quad |\mathring{g}^{ti} - g^{ti}| \leq \frac{C}{|x|} \frac{1}{\cosh(x)} \quad \text{and} \quad |\mathring{g}^{ij} - g^{ij}| \leq \frac{C}{|x|} \quad (\text{A.4})$$

for  $i, j \neq t$ . Note that these imply

$$|g^{tt}| \leq \frac{C}{\cosh^2(x)}, \quad |g^{ti}| \leq \frac{C}{|x|} \frac{1}{\cosh(x)}, \quad |g^{ii}| \leq C \quad \text{and} \quad |g^{ij}| \leq \frac{C}{|x|} \quad \text{for } i \neq j. \quad (\text{A.5})$$

Regarding the Christoffel symbols we note that

$$\Gamma_{ij}^l = \frac{1}{2} g^{lk} (\partial_j g_{ki} + \partial_i g_{kj} - \partial_k g_{ij}).$$

Since only  $\partial_x \mathring{g}_{tt}$ ,  $\partial_\theta \mathring{g}_{\phi\phi}$  are non-zero, the estimates of all Christoffels not containing either of those derivatives follow from (A.5) and  $|\partial_k g_{ij}| \leq \frac{C \cosh^{\#t}(x)}{|x|}$  for  $(k, i, j) \neq (t, x, x), (\theta, \phi, \phi)$ . So we have

$$|\Gamma_{ij}^k| \leq \frac{C}{|x|} \cosh^{\#t}(x) \quad (\text{A.6})$$

if  $(k, i, j) \neq (t, t, x), (x, t, t), (\phi, \phi, \theta), (\theta, \phi, \phi)$ . The remaining Christoffels are  $\Gamma_{tx}^t$ ,  $\Gamma_{tt}^x$ ,  $\Gamma_{\phi\theta}^\phi$  and  $\Gamma_{\phi\phi}^\theta$ . For these, the summands appearing in  $|\mathring{\Gamma} - \Gamma|$  for which the  $\mathring{g}$ -part does not vanish can be estimated using  $|\partial_x \mathring{g}_{tt} - \partial_x g_{tt}| \leq \frac{C}{|x|} |\partial_x \mathring{g}_{tt}| \leq \frac{C}{|x|} \cosh^2(x)$  (since  $\cosh$  and  $\sinh$  have the same behaviour at infinity) and  $|\partial_\theta \mathring{g}_{\phi\phi} - \partial_\theta g_{\phi\phi}| \leq \frac{C}{|x|} |\partial_\theta \mathring{g}_{\phi\phi}| \leq \frac{C}{|x|}$  by (2.6), (A.4) and (A.1). This gives

$$|\mathring{\Gamma}_{ij}^k - \Gamma_{ij}^k| \leq \frac{C}{|x|} \cosh^{\#t}(x) \quad \text{and} \quad |\mathring{\Gamma}_{ij}^k| \leq C \cosh^{\#t}(x). \quad (\text{A.7})$$

for these four Christoffels.

For the components  $R_{iklm}$  of the Riemann tensor we use

$$R_{iklm} = \frac{1}{2} (\partial_k \partial_l g_{im} + \partial_i \partial_m g_{kl} - \partial_k \partial_m g_{il} - \partial_i \partial_l g_{km}) + g_{np} (\Gamma_{kl}^n \Gamma_{im}^p - \Gamma_{km}^n \Gamma_{il}^p). \quad (\text{A.8})$$

Again, if those products always contain at least one factor that is zero for  $\mathring{g}$ , the desired estimates follows easily from the assumption on  $\partial^2 h$ ,  $h$  and (A.6), (A.7). The remaining two cases are  $R_{xtxt}$  and  $R_{\theta\phi\theta\phi}$  where  $\mathring{g}_{tt} \left( \mathring{\Gamma}_{tx}^t \mathring{\Gamma}_{xt}^t - \mathring{\Gamma}_{tt}^t \mathring{\Gamma}_{xx}^t \right) = \mathring{g}_{tt} (\mathring{\Gamma}_{tx}^t)^2 = \sinh^2(x)$  and  $\mathring{g}_{\phi\phi} \left( \mathring{\Gamma}_{\phi\theta}^\phi \mathring{\Gamma}_{\theta\phi}^\phi - \mathring{\Gamma}_{\phi\phi}^\phi \mathring{\Gamma}_{\theta\theta}^\phi \right) = \mathring{g}_{\phi\phi} (\mathring{\Gamma}_{\phi\theta}^\phi)^2 = \cos^2(\theta)$ , respectively. For these cases we again use A.1 (and that  $\sinh$  and  $\cosh$  behave the same at infinity and that in our charts  $\sin(\theta)$  is bounded away from zero).

Finally, the asymptotics for  $\text{Ric}$  and  $R$  follow from (A.4), (A.5) and the asymptotics of  $R_{iklm}$  using the same arguments.

## B Weakening of the null energy condition

In this appendix we wish to indicate how the results of this paper as summarized in Theorems 1.1 and 1.2 continue to hold under the weaker integrated curvature condition (1.1).

The NEC enters into the proof of Theorem 1.1 in two ways:

- (i) It is used in the proof of Lemma 2.13.
- (ii) It is used in results such as Theorem 3.9 which rely on the ‘null splitting theorem’, Theorem IV.1 in [4].

The following is sufficient to ensure that Lemma 2.13 holds under the curvature condition (1.1).

**Proposition B.1.** *Assume  $(M, g)$  satisfies the curvature condition (1.1). If  $\eta : (-\infty, \infty) \rightarrow M$  is a complete null line then  $\text{Ric}(\eta'(s), \eta'(s)) = 0$  for all  $s \in \mathbb{R}$ .*

*Proof.* This follows almost immediately from Corollary 3.3 in [2]. Since  $\eta$  is a complete null line, it is free of conjugate points. Then, by [2, Corollary 3.3],

$$\int_{-\infty}^{\infty} \text{Ric}(\eta'(s), \eta'(s)) ds \leq 0.$$

But then the curvature condition (1.1) implies that we have equality in the above. In this case, [2, Corollary 3.3] further implies that  $\text{Ric}(\eta'(s), \eta'(s)) = 0$  for all  $s \in \mathbb{R}$ .  $\square$

The NEC enters into the proof of [4, Theorem IV.1] in only one place, namely through Lemma IV.2. The following proposition shows that this lemma remains valid under the curvature condition (1.1).

**Proposition B.2.** *Suppose  $S$  is an achronal  $C^0$  future null hypersurface in  $(M, g)$  whose null generators are future geodesically complete. If along each null generator  $\eta : [0, \infty) \rightarrow \mathbb{R}$  the Ricci curvature satisfies (1.1) then  $S$  has null mean curvature  $\theta \geq 0$  in the sense of support hypersurfaces.*

We refer the interested reader to [4] for the definitions of terms being used in the statement of this proposition. The proof makes use of the following lemma which is proved in [3, Section 3].

**Lemma B.3.** *Consider the initial value problem*

$$\begin{aligned} x'' + p(s)x &= 0 \\ x(0) &= 1 \\ x'(0) &= a \end{aligned} \tag{B.1}$$

*If  $p \in C^\infty([0, \infty))$  satisfies*

$$\int_0^\infty p(s) ds > a \tag{B.2}$$

*then the unique solution to (B.1) has a zero on  $[0, \infty)$ .*



*Proof of Proposition B.2.* Given  $p \in S$ , let  $\eta : [0, \infty) \rightarrow \mathbb{R}$  be a null generator of  $S$  starting at  $\eta(0) = p$ . For any  $\epsilon > 0$ , we have,

$$\int_0^\infty \text{Ric}(\eta'(s), \eta'(s)) ds > -(n-2)\epsilon. \quad (\text{B.3})$$

By the lemma, the unique solution  $x = x(s)$  to the initial value problem (B.1), with

$$p(s) = \frac{1}{n-2} \text{Ric}(\eta'(s), \eta'(s)) \quad \text{and} \quad a = -\epsilon \quad (\text{B.4})$$

satisfies  $x(r_*) = 0$  for some  $r_* \in (0, \infty)$ . We may assume  $r_*$  is the first zero of  $x(s)$ .

Fix  $r > r_*$ . As in the proof of [4, Lemma IV.2], by considering  $\partial J^-(\eta(r))$  we obtain a smooth null hypersurface  $S_r$  defined in a neighborhood of  $\eta|_{[0,r]}$  such that  $S_r$  is a past support hypersurface for  $S$  at  $p$ .

Let  $\theta = \theta(s)$  be the null expansion of  $S_r$  along  $\eta|_{[0,r]}$ ;  $\theta$  satisfies the Raychaudhuri equation [4, (II.4)]. Let  $y = y(s)$  be defined by the substitution,

$$\frac{y'}{y} = \frac{1}{n-2} \theta(s)$$

with  $y(0) = 1$ . A standard computation shows that  $y$  satisfies the IVP (B.1) with

$$p(s) = \frac{1}{n-2} (\text{Ric}(\eta', \eta') + \sigma^2) \quad \text{and} \quad a = \frac{1}{n-2} \theta(p). \quad (\text{B.5})$$

Suppose  $\theta(0) < -(n-2)\epsilon$ . By a basic ODE comparison result we have  $y(s) \leq x(s)$  (up to the first zero of  $y$ ), where  $x(s)$  is the solution to (B.1)+(B.4). In particular  $y(s)$  must go to zero somewhere on  $[0, r_*]$ . This implies that  $\theta$  is not defined everywhere on this interval, which is a contradiction since  $\theta = \theta(s)$  is smooth on  $[0, r]$ . Thus we must have  $\theta(0) \geq -(n-2)\epsilon$ . Since  $\epsilon$  is arbitrary, this proves the proposition.  $\square$

With regard to Theorem 1.2, the additional arguments of Section 4, beyond those of Section 3, show that it is sufficient for the NEC,  $\text{Ric}(X, X) \geq 0$ , to hold for vectors  $X$  tangent to null rays. But this follows trivially from (1.1), since, under the assumption that Ric is covariant constant, the integrand is constant.

## References

- [1] P. T. Chruściel, E. Delay, G. J. Galloway, and R. Howard, *Regularity of horizons and the area theorem*, Ann. Henri Poincaré **2** (2001), no. 1, 109–178.
- [2] P. E. Ehrlich and S.-B. Kim, *From the Riccati inequality to the Raychaudhuri equation*, Differential geometry and mathematical physics (Vancouver, BC, 1993), Contemp. Math., vol. 170, Amer. Math. Soc., Providence, RI, 1994, pp. 65–78.

- [3] G. J. Galloway, *Some results on the occurrence of compact minimal submanifolds*, Manuscripta Math. **35** (1981), no. 1-2, 209–219.
- [4] ———, *Maximum principles for null hypersurfaces and null splitting theorems*, Ann. Henri Poincaré **1** (2000), no. 3, 543–567.
- [5] G. J. Galloway, K. Schleich, D. M. Witt, and E. Woolgar, *Topological censorship and higher genus black holes*, Phys. Rev. D (3) **60** (1999), no. 10, 104039, 11.
- [6] H. K. Kunduri and J. Lucietti, *Classification of near-horizon geometries of extremal black holes*, Living Reviews in Relativity **16** (2013), no. 1, 8.
- [7] O. Lunin, *Bubbling geometries for  $AdS_2 \times S^2$* , J. High Energy Phys. (2015), no. 10, 167, front matter+64.
- [8] J. Maldacena, *The large  $N$  limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998), no. 2, 231–252.
- [9] ———, 2017, personal communication.
- [10] J. Maldacena, J. Michelson, and A. Strominger, *Anti-de Sitter fragmentation*, J. High Energy Phys. (1999), no. 2, Paper 11, 23.
- [11] E. Minguzzi and M. Sánchez, *The causal hierarchy of spacetimes*, Recent developments in pseudo-Riemannian geometry, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008, pp. 299–358. MR 2436235
- [12] R. Ponge and H. Reckziegel, *Twisted products in pseudo-Riemannian geometry*, Geom. Dedicata **48** (1993), no. 1, 15–25.
- [13] J. M. M. Senovilla, *Second-order symmetric Lorentzian manifolds. I. Characterization and general results*, Classical Quantum Gravity **25** (2008), no. 24, 245011, 25.
- [14] F. J. Tipler, *General relativity and conjugate ordinary differential equations*, J. Differential Equations **30** (1978), no. 2, 165–174.
- [15] H. Wu, *On the de Rham decomposition theorem*, Illinois J. Math. **8** (1964), 291–311.