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Final Report: Rational and Algebraic Solutions of Algebraic Ordinary Differential Equations

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Abstract

The main aim of this report is to study new algorithms for determining rational and algebraic solutions of first-order algebraic ordinary differential equations (AODEs). The problem of determining closed form solutions of first-order AODEs has a long history, and it still plays a role in many branches of mathematics. Our interests are algebraic general solutions and rational general solutions. We approach first-order AODEs from algebraic geometric aspects. By considering the derivative as a new indeterminate, a first-order AODE can be viewed as a hypersurface over the ground field. Therefore tools from algebraic geometry are applicable. In particular, we use birational transformations of algebraic hypersurfaces to transform the differential equation to another one for which we hope that it is easier to solve. This geometric approach leads us to a procedure for determining an algebraic general solution of a parametrizable first-order AODE. A general solution contains an arbitrary constant. For the problem of determining a rational general solution in which the constant appears rationally, we propose a decision algorithm for the general class of first-order AODEs.

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Contents

1 Introduction

A first-order AODE is a differential equation of the form $F(x, y, y') = 0$, where F is a polynomial in three variables with coefficients in an algebraically closed field, for instance $\overline{\mathbb{Q}}$, the field of algebraic numbers. Solving the differential equation is the problem of determining differentiable functions $y = y(x)$ such that $F(x, y(x), y'(x)) = 0$. If $y(x)$ is an algebraic (rational, polynomial) functions, then it is called an algebraic (rational, polynomial, respectively) solution. A solution may contain an arbitrary constant. Such a solution is called a general solution. For example, $y(x) = x^2 + c$ is a general solution of the differential equation $y' - 2x = 0$. Solving first-order AODEs is a fundamental problem in the theory of (non-linear) algebraic differential equations.

First-order AODEs have been studied a lot and there are many solution methods for special classes of such ODEs. The study of these ODEs can be dated back to the work of Fuchs [10], and later by Poincaré [25]. In [20], Malmquist studied the class of first-order AODEs having transcendental meromorphic solutions, and Eremenko revisited later in [7]. In the 1970s, Matsuda classified differential function fields having no movable critical points up to isomorphism of differential fields [21]. The theory by Matsuda brings modern wind to the algebraic theory of first-order AODEs. Following this direction, Eremenko presented a theoretical consideration on a degree bound for rational solutions [8].

The problem of finding closed form solutions of first-order AODEs has been considered widely in the literature. Among non-linear first-order AODEs, Riccati equations can be considered as the simplest ones. In [19], Kovacic solved completely the problem of computing Liouvilian solutions of a second order linear ODE with rational function coefficients. In the process, Kovacic also proposed an algorithm for determining all rational solutions of a Riccati equation. Solving a first-order first-degree AODE is a much harder problem. The problem of determining an algebraic general solution for a first-order first-degree AODE is one of an equivalent version of the Poincaré problem. This problem is still open. In [5], Carnicer investigated a degree bound for algebraic solutions for first-order first-degree AODEs in non-dicritical cases. Hubert [16] found implicit solutions by computing Gröbner bases.

The problem of studying symbolic solutions for first-order AODEs from an algebro-geometric approach has received much attention in the last decade. Algorithms for the class of first-order autonomous AODEs has been proposed by Feng and Gao [9, 1]. The algorithm is based on the fact that by considering the derivative as a new indeterminate, the differential equation can be viewed an algebraic curve. Applying this idea to the general class of first-order AODEs, and combining it with Fuchs' theorem on first-order AODEs without movable critical points, Chen and Ma [6] presented an algorithm for determining a special class of rational general solutions. However, their algorithm is incomplete due to two reasons: the necessary condition for the existence of the solution is not proven to be algorithmically checkable, and a good rational parametrization is required in advance. Ngô and Winkler [22, 24, 23] applied the algebro-geometric approach to general non-autonomous first-order AODEs. Using parametrization of algebraic surfaces, they associate to the given parametrizable AODE an associated system of algebraic equations in the parameters. This associated system is a planar rational system. In order to complete the algorithm, a degree bound for irreducible invariant algebraic curves of the planar rational system is required. The problem of finding a uniform bound for the degree of invariant algebraic curves for planar rational systems is known as the Poincaré problem. This difficult problem has been solved by Carnicer [5], but only generically for the non-dicritical case. So the algorithm of Ngô and Winkler, although producing general rational solutions in almost all situations where such a solution exists, is still no complete decision algorithm. Following this direction, a generalization to the class of higher order AODEs [15], and even to algebraic partial differential equations [11] is presented. So far no general algorithm for deciding the existence and, in the positive case, computing an algebraic/rational general solution, and all particular rational solutions exists.

In this report, we present:

1. A procedure for determining an algebraic general solution of a parametrizable first-order AODE (see Algorithm 2).

2. A full algorithm for determining a rational general solution, in which the constant appears rationally, for a general first-order AODE (see Algorithm 4).

This generalizes the works by Feng and Gao [9], Chen and Ma [6], Ngô and Winkler [22, 24, 23].

In Section 2, we recall basic notations from differential algebra and algebraic geometry. In Section 3, we approach first-order AODEs from an algebraic geometric aspect. By considering the derivative as a new indeterminate, a given first-order AODE can be seen as an algebraic equation. This algebraic equation defines an algebraic surface over the ground field. An algebraic solution of the differential equation corresponds to an algebraic curve on the surface which satisfies certain condition. Therefore tools from algebraic geometry are applicable. In particular, birational transformation of algebraic surfaces is used to transform the differential equation to a planar rational system. The key point is that there is a faithful relation between algebraic general solutions of the given differential equation and algebraic general solutions of the planar rational system. Solving a general planar rational system is still a very hard problem. But in many cases, the obtained planar rational system is easy to solve. This algebraic geometric approach leads to a procedure for determining an algebraic general solution for first-order AODEs.

A similar method is presented in Section 4. By considering the derivation as a new indeterminate, we can also view the differential equation as an algebraic equation which defines an algebraic curve over the field of rational functions over the ground field. With a similar process, birational transformation of algebraic curves is used to transform the given differential equation to a first-order first-degree AODE. We prove that optimal parametrization of algebraic curves over the field of rational functions can be achieved within the field of rational functions. This guarantees us to do the process in a controllable way. Consequently, a decision algorithm for determining a rational general solution for which the constant appears rationally of a first-order AODE is established.

2 Preliminaries

In this section, we briefly recall some basic notions in differential algebra and algebraic geometry. This section should not be considered as an introduction to differential algebra and parametrization of hypersurface. For further detail, we refer the reader to [18, 27] for differential algebra, and to [31, 28] for parametrization of algebraic curves and surfaces.

2.1 General solutions of AODEs

From now on by $\mathbb{K}$ we denote a computational algebraically closed field of characteristic zero with the trivial derivation. In practice, we might choose $\mathbb{K} = \overline{\mathbb{Q}}$ the field of algebraic numbers. All derivatives are understood as the usual ones.

An algebraic ordinary differential equation is a differential equation of the form

$$F(x, y, y', ..., y^{(n)}) = 0 \tag{1}$$

where $F \in \mathbb{K}[x]\{y\} \backslash \mathbb{K}[x][y]$. Without lost on generality, we can always assume that F is an irreducible polynomial in $\mathbb{K}[x, y, y', ..., y^{(n)}]$. Otherwise F can be factored as the product of irreducible factors. In this case, the set of solutions of the given differential equation is equal to the union of the sets of solutions of AODEs which are defined by the irreducible factors of F.

The notion "general solution" of an AODEs can be described from differential algebra context. In general a general solution of an AODE is defined as a generic zero of a certain associated prime differential ideal. In our situation such ideal much be established from F. It is well-known that, in a polynomial ring in finitely many variables over a field, a principle ideal generated by an irreducible polynomial is prime. It is no longer true in the case of differential polynomial rings. In particular, neither $[F]$ nor $\{F\}$ is a prime differential ideal of $\mathbb{K}(x)\{y\}$, even if F is an irreducible polynomial. Fortunately, Ritt proved that:

Lemma 2.1 (see [27])**.** *Let $F \in \mathbb{K}(x)\{y\}$ such that F is an irreducible polynomial in $\mathbb{K}[x, y, y', ..., y^{(n)}]$. Then the ideal $\{F\}$ can be factored as:*

$$\{F\} = (\{F\} : S_F) \cap \{F, S_F\}$$

where $(\{F\} : S_F) := \{G \in \mathbb{K}(x)\{y\} \,|\, G.S_F \in \{F\}\}$ is a prime differential ideal.

The lemma shows that the ideal $(\{F\} : S_F)$ is the unique essential component (among finitely many essential components of $\{F\}$) that does not contain the separant S_F of F. On the other hand the second component $\{F, S_F\}$ is the intersection of the other essential components of $\{F\}$. It leads us to the definition of general solutions of an AODE.

Definition 2.2. Consider the differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$.

i. A zero of the radical ideal $\{F\}$ is called a solution of the differential equation.

ii. A generic zero of the differential ideal $(\{F\} : S_F)$ is called a general solution.

iii. A zero of the ideal $\{F, S_F\}$ is called a singular solution.

Definition 2.3. Consider the differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$, let ξ be a solution which is contained in a differential field L extended from $\mathbb{K}(x)$. We denote by K the field of constants of L.

i. ξ is called an algebraic solution if there is a non-zero polynomial $G \in K[x, y]$ such that $G(x, \xi) = 0$.
In this case, G is called an annihilating polynomial of ξ.

ii. If furthermore $\deg_y G = 1$, then ξ is called a rational solution.

iii. ξ is called an algebraic (resp. rational, polynomial) general solution if it is a general solution and algebraic (resp. rational, polynomial).

Given an algebraic solution ξ of the differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$, there are infinitely many corresponding annihilating polynomials. If we ask for irreducible polynomials among them, then there is only one up to multiplying by a non-zero constant. If $G(x, y)$ is an irreducible annihilating polynomial of ξ, then all root $y = y(x)$ of the algebraic equation $G(x, y) = 0$ are solutions of the differential equation, (see [1], Lemma 2.4). Therefore, by abuse of notation, G is sometimes called a solution.

The following lemma is a well-know criteria for checking whether a solution is general (see [27]).

Lemma 2.4. *A solution ξ of the differential equation*

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

is a general solution if and only if

$$\forall H \in k(x)\{y\},\ H(\xi) = 0 \Rightarrow \mathit{prem}(H, F) = 0$$

Proposition 2.5. *Let ξ be an algebraic solution of the differential equation*

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

with the irreducible annihilating polynomial G. If ξ is a general solution, then at least one of the coefficients of G contain a constant which is transcendental over $\mathbb{K}$.

Proof. By contradiction, if $G(x, y) \in \mathbb{K}[x, y]$, then $G = \text{prem}(G, F) \neq 0$. It is contradiction with the fact that $G(x, \xi) = 0$. Thus at least one of the coefficients of G is a constant which is not in $\mathbb{K}$. Since $\mathbb{K}$ is algebraically closed, such constant is transcendental over $\mathbb{K}$. □

From the previous proposition, an algebraic general solution can be viewed as a class of algebraic solutions which is parametrized by a certain number of parameters. In particular if $y^{(n)}$ is the highest derivation appearing on F, then the number of independent parameters needed to paramatrized a general solution is exactly n (see [18, Thm. 6, Sec. 12, Chp. 2]).

2.2 Parametrization of algebraic curves and surfaces

Consider a first-order AODE, $F(x, y, y') = 0$, for an irreducible non-constant polynomial F. We view the equation to be an algebraic one by replacing the derivative by an independent variable, i. e. $F(x, y, z) = 0$. Depending on the ground field the zero set of such an equation defines an algebraic curve or an algebraic surface.

$$\mathcal{C} = \left\{(a_1, a_2) \in \mathbb{A}^2(\overline{\mathbb{K}(x)}) \mid F(x, a_1, a_2) = 0\right\} ,$$
$$\mathcal{S} = \left\{(a_0, a_1, a_2) \in \mathbb{A}^3(\mathbb{K}) \mid F(a_0, a_1, a_2) = 0\right\} .$$

For higher dimensional spaces such zero sets of single polynomials are called hypersurfaces.

Definition 2.6. The algebraic curve $\mathcal{C}$ is called the *corresponding curve*. The algebraic surface $\mathcal{S}$ is called the *corresponding surface*.

By definition the algebraic curves and surfaces are given implicitly by the defining equation. Very often it is useful to have a parametric expression for the points on the curve or surface.

Definition 2.7. A *rational parametrization*, or briefly, a parametrization of a curve $\mathcal{C}$ over $\mathbb{A}^2(\mathbb{K})$ is a rational map $\mathcal{P} : \mathbb{A}^1(\mathbb{K}) \to \mathcal{C} \subseteq \mathbb{A}^2(\mathbb{K})$ such that the image of $\mathcal{P}$ is dense in $\mathcal{C}$ (with respect to the Zariski topology).

Similarly a (rational) parametrization of a surface $\mathcal{S}$ over $\mathbb{A}^3(\mathbb{K})$ is a rational map $\mathcal{P} : \mathbb{A}^2(\mathbb{K}) \to \mathcal{S} \subseteq \mathbb{A}^3(\mathbb{K})$ such that the image of $\mathcal{P}$ is dense in $\mathcal{S}$.

If, furthermore, $\mathcal{P}$ is a birational equivalence, $\mathcal{P}$ is called a *proper* parametrization.

A parametrization is called optimal, if the degree of its coefficient field is minimal (see [31] for further details).

Let $\mathcal{P}_{\mathcal{C}}(t) = (p_1(x, t), p_2(x, t))$ be a parametrization over $\overline{\mathbb{K}(x)}$ of the corresponding curve of an AODE. Then $\mathcal{P}_{\mathcal{S}}(s_1, s_2) = (s_1, p_1(s_1, s_2), p_2(s_1, s_2))$ is an algebraic parametrization of the corresponding surface. If $\mathcal{P}_{\mathcal{C}}$ is rational in x then $\mathcal{P}_{\mathcal{S}}$ is a rational parametrization. However, there are first-order AODEs which admit a rational parametrization of the corresponding surface but not of the corresponding curve. Consider for instance the

AODE, $F(x, y, y') = y'^2 - y^3 - x^2 = 0$. The corresponding curve has genus 1, whereas the corresponding surface can be parametrized by $\left(\frac{s(1-s^2)}{t^3}, \frac{1-s^2}{t^2}, \frac{1-s^2}{t^3}\right)$.

It is well-known that if an algebraic curve or surface admits a rational parametrization, then it admits a proper parametrization. In the affirmative case, for curves one can compute such a proper parametrization with optimal coefficient field. For more details on rationality we refer to [31] and [35, 32, 28] for curves and surfaces respectively.

Theorem 2.8 (Rationality Criterion)**.** *An algebraic curve admits a rational parametrization if and only if its genus is equal to zero.*

An algebraic surface admits a rational parametrization if and only if both its arithmetic genus and the second plurigenus are equal to zero.

Furthermore, there is a relation between different proper parametrizations of curves and surfaces respectively.

Lemma 2.9. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two proper parametrizations of some algebraic hypersurface. Then there exists a rational function R such that $\mathcal{Q} = \mathcal{P}(R)$.*

- *In case of curves, R is a Möbius transformation, i. e. a linear rational function $R(s_1) = \frac{a_0 + a_1 s_1}{b_0 + b_1 s_1}$ with $a_0 b_1 - a_1 b_0 \neq 0$.*
- *In case of surfaces, R is a Cremona transformation, i. e. a birational map of the plane to itself, and hence by the Theorem of Castelnuovo-Noether a finite composition of quadratic transformations and projective linear transformations (c. f. [32, 35]).*

Definition 2.10. A point A on the corresponding curve $\mathcal{C}$ is called an *algebraic solution point* if its coordinates have the form $(y(x), y'(x))$ for some $y(x) \in \overline{\mathbb{K}(x)}$. If furthermore $y(x) \in \mathbb{K}(x)$, A is called a *rational solution point*.

Finding an algebraic/rational general solution of $F(x, y, y') = 0$ is reduced to looking for a class of algebraic/rational solution points $(y(x), y'(x))$ which depend on a parameter c.

3 Algebraic general solutions of first-order AODEs

This section is based on the author's works in [33]. In this section we present a procedure for determining an algebraic general solution of a first-order AODE. In order to use the technique of rational parametrization, we add an additional assumption to the initial differential equation, that the algebraic equation obtained when we replace the derivation y' by a new indeterminate defines a rational surface. A first-order AODE satisfying this additional assumption is called surface-parametrizable. The general schedule for determining an algebraic general solution of a surface-parametrizable first-order AODE is as follows.

We associate for each surface-parametrizable first-order AODE a planar rational system, which is so called the associated differential system. The key observation is that algebraic general solutions of the initial differential equation can be determined faithfully from an algebraic general solution of the associated differential system (see Section 3.1). This step is inherited from the work by Ngô and Winkler in [22].

The problem of determining an algebraic general solution of a surface parametrizable first-order AODE is now reduced to the problem of computing an algebraic general solution of a planar rational system. The latter problem is hard in general. But in case a rational first integral is provided, or even only a degree bound for a rational first integral is given, we propose an algorithm to determine an algebraic general solution (see Section 3.2). Finally, if a surface-parametrizable first-order AODE is given together with a degree bound for an algebraic general solution, we can compute an algebraic general solution explicitly (see Section 3.3).

3.1 Associated Differential System

In this section, we construct for each surface-parametrizable first-order AODE a planar rational system. Although the construction is as similar as the one described in Ngô and Winkler [22], it is briefly summarized here for self-containedness. Several facts relating to their algebraic general solutions are investigated.

Let us first give a formal definition for surface-parametrizable first-order AODE.

Definition 3.1. A first-order AODE $F(x, y, y') = 0$ is called surface parametrizable if its corresponding surface, say $\mathcal{S}$, in $\mathbb{A}^2(\mathbb{K})$ defined by $F(x, y, z) = 0$ is rational.

In the other words, there is a rational map $\mathcal{P} : \mathbb{A}^2_{\mathbb{K}} \to \mathcal{S} \subset \mathbb{A}^3_{\mathbb{K}}$ defined by $\mathcal{P}(s, t) := (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t))$ for some rational functions $\chi_1, \chi_2, \chi_3 \in \mathbb{K}(s, t)$ such that $F(\mathcal{P}(s, t)) = 0$, and $\mathcal{P}$ is invertible. Such $\mathcal{P}$ is called a *proper parametrization* of the surface $\mathcal{S}$. Algorithm for determining a parametrization of a rational surface is investigated, for instance, there is one in [28]. During this section and so on, we always assume that a surface parametrizable first-order AODE is equipped with a proper parametrization $\mathcal{P}$.

Now let us fix an algebraic general solution $\xi = \xi(x)$ of the surface parametrizable first-order AODE $F(x, y, y') = 0$. Then $F(x, \xi(x), \xi'(x)) = 0$. Denote $(s(x), t(x)) := \mathcal{P}^{-1}(x, \xi(x), \xi'(x))$, a representation of the inverse of $(x, \xi(x), \xi'(x))$ via $\mathcal{P}$. Since $\mathcal{P}$ is proper, $(s(x), t(x))$ is a pair of algebraic functions satisfying $\mathcal{P}(s(x), t(x)) = (x, \xi(x), \xi'(x))$. Therefore

$$\begin{cases} \chi_1(s(x), t(x)) = x \\ \chi_2'(s(x), t(x)) = \chi_3(s(x), t(x)) \end{cases}$$

Differentiating both sides of the first equation, and expanding the second one gives us a linear system on $s'(x)$ and $t'(x)$.

$$\begin{cases} s'(x)\dfrac{\partial}{\partial s}\chi_1(s(x),t(x)) + t'(x)\dfrac{\partial}{\partial t}\chi_1(s(x),t(x)) = 1 \\ s'(x)\dfrac{\partial}{\partial s}\chi_2(s(x),t(x)) + t'(x)\dfrac{\partial}{\partial t}\chi_2(s(x),t(x)) = \chi_3(s(x),t(x)) \end{cases}$$

Since $\mathcal{P}$ is a birational equivalent, the Jacobian matrix

$$\begin{bmatrix} \frac{\partial \chi_1}{\partial s} & \frac{\partial \chi_2}{\partial s} & \frac{\partial \chi_3}{\partial s} \\ \\ \frac{\partial \chi_1}{\partial t} & \frac{\partial \chi_2}{\partial t} & \frac{\partial \chi_3}{\partial t} \end{bmatrix}$$

has generic rank 2. Without lost on general, we can always assume that the determinant

$$\begin{vmatrix} \frac{\partial \chi_1}{\partial s} & \frac{\partial \chi_2}{\partial s} \\ \\ \frac{\partial \chi_1}{\partial t} & \frac{\partial \chi_2}{\partial t} \end{vmatrix}$$

is non-zero. Furthermore, we claim that $g(s(x),t(x)) \neq 0$. It will be asserted by the following lemma.

Lemma 3.2. *With notation as above. Then*

$$\forall R \in \mathbb{K}(s,t),\ \ R(s(x),t(x)) = 0 \Rightarrow R = 0$$

Now $s'(x)$ and $t'(x)$ can be solved by Cramer's rule from the linear system. Thus $(s(x),t(x))$ is an algebraic solution of the planar rational system:

$$\begin{cases} s' = \dfrac{\chi_3(s,t)\frac{\partial}{\partial t}\chi_1(s,t) - \frac{\partial}{\partial t}\chi_2(s,t)}{g(s,t)} \\ t' = \dfrac{\frac{\partial}{\partial s}\chi_2(s,t) - \chi_3(s,t)\frac{\partial}{\partial s}\chi_1(s,t)}{g(s,t)} \end{cases} \tag{2}$$

Definition 3.3. System (2) is called the associated differential system of the differential equation $F(x,y,y') = 0$ with respect to the proper parametrization $\mathcal{P}$.

We summary here the result of the construction of the associated system above.

Theorem 3.4. *Let $F(x,y,y') = 0$ be a surface parametrizable first-order AODE and consider its associated differential system (2) with respect to a given proper parametrization $\mathcal{P}$. If $y = y(x)$ is an algebraic general solution of the differential equation $F(x,y,y') = 0$, then*

$$(s(x),t(x)) := \mathcal{P}^{-1}(x,y(x),y'(x))$$

is an algebraic general solution of the associated system.

Proof. $(s(x), t(x))$ is an algebraic solution of the associated system as we established. Lemma 3.2 asserts that it is in fact a general solution. □

Theorem 3.5. *Let $F(x, y, y') = 0$ be a surface parametrizable first-order AODE and consider its associated system (2) with respect to a given proper parametrization $\mathcal{P}$. If $(s(x), t(x))$ is an algebraic general solution of the associated system, then*

$$y(x) := \chi_2 \left(s\left(2x - \chi_1(s(x), t(x))\right), t\left(2x - \chi_1(s(x), t(x))\right)\right)$$

is an algebraic general solution of the differential equation $F(x, y, y') = 0$.

Proof. As in the construction, $s(x), t(x)$ must satisfies the following system:

$$\begin{cases} \chi_1'(s(x), t(x)) = 1 \\ \chi_2'(s(x), t(x)) = \chi_3(s(x), t(x)) \end{cases}$$

The first relation yields $c := \chi_1(s(x), t(x)) - x$ is an arbitrary constant. Thus we have

$$\begin{cases} \chi_1(s(x-c), t(x-c)) = x \\ \chi_2(s(x-c), t(x-c)) = y(x) \\ \chi_3(s(x-c), t(x-c)) = y'(x) \end{cases}$$

Therefore $y(x)$ is an algebraic general solution of $F(x, y, y') = 0$.

It remains to prove that $y(x)$ is a general solution. To this end, let arbitrary $G \in \mathbb{K}(x)\{y\}$ such that $G(y(x)) = 0$. Since F is of order 1, $\text{prem}(G, F) \in \mathbb{K}(x)[y, y']$. Let $R \in \mathbb{K}, y, y']$ be the numerator of $\text{prem}(G, F)$. Then $R(x, y(x), y'(x)) = 0$. It implies $R(\mathcal{P}(s(x-c), t(x-c))) = 0$. Since c can be chosen arbitrary, we have $R(\mathcal{P}(s(x), t(x))) = 0$. Now, applying the lemma 3.2 yields $R(\mathcal{P}(s, t)) = 0$. So that $R(x, y, z) = R(\mathcal{P}(\mathcal{P}^{-1}(x, y, z))) = 0$. It follows $\text{prem}(G, F) = 0$. Hence $y(x)$ is a general solution. □

The previous two theorems establish a one-to-one correspondence between algebraic general solutions of a paramatrizable first-order AODE and algebraic general solutions of its associated system which is a planar rational system. Furthermore the correspondence is formulated explicitly. Once an algebraic general solution of its associated system is known, the corresponding algebraic general solution of the given surface parametrizable first-order AODE can be determined immediately. The problem of finding an algebraic general solution of paramatrizable first-order AODEs can be reduced to the problem of determining an algebraic general solution of a planar rational system.

Is is important to notice that the one-to-one correspondence holds not only for the class of algebraic general solutions, but also for the general class of general solutions (which do not necessary satisfy the property of being algebraic functions). By just repeating the above process without assumption that the general solutions are algebraic, we obtain:

Theorem 3.6. *Let $F(x, y, y') = 0$ be a surface parametrizable first-order AODE and consider its associated system (2) with respect to a given proper parametrization $\mathcal{P}$.*

i. If $y(x)$ is a general solution of the differential equation $F(x, y, y') = 0$, then

$$(s(x), t(x)) := \mathcal{P}^{-1}(x, y(x), y'(x))$$

is a general solution of the associated system.

ii. If $(s(x), t(x))$ is a general solution of the associated system, then

$$y(x) := \chi_2(s(2x - \chi_1(s(x), t(x))), t(2x - \chi_1(s(x), t(x))))$$

is a general solution of the given differential equation.

3.2 Planar rational system and its algebraic general solutions

This section is devoted to the problem of computing explicitly an algebraic general solution of the planar rational system. Whereas the problem of finding explicit algebraic solutions of planar rational systems has received only little attention in the literature, the problem of finding implicit algebraic solutions, or in the other words, finding irreducible invariant algebraic curves and rational first integral, has been heavily studying. By combining these results and the idea for finding algebraic general solutions of autonomous first-order AODEs of Aroca et. al. (see [1]), we will present an algorithm for determining an algebraic general solution of a planar rational system with a given rational first integral.

Definition 3.7. A planar rational system is a differential system of order 1 of the form:

$$\begin{cases} s' = M(s, t) \\ t' = N(s, t) \end{cases} \tag{3}$$

where M, N are rational functions on s, t with coefficients in $\mathbb{K}$.

If M, N are polynomials, it is called a planar polynomial system.

Given a planar rational system, we are interested in its algebraic solutions. The key objects to investigate information about algebraic solutions of a planar rational system are invariant algebraic curves and rational first integrals. Our plan is first using known algorithms to find a rational first integral of the system, and then its irreducible algebraic curves. Secondly, each irreducible algebraic curve will derive the system into two autonomous first-order AODEs which can be solved explicitly by the procedure of Aroca et. al. (see [1]).

Definition 3.8. An algebraic curve defined by $G(s,t)=0$ is called an *invariant algebraic curve* of the planar rational system

$$\begin{cases} s' = M(s,t) \\ t' = N(s,t) \end{cases} \tag{4}$$

where M, N are rational functions on s, t with coefficients in $\mathbb{K}$, if

$$M_1 N_2 \frac{\partial G}{\partial s} + M_2 N_1 \frac{\partial G}{\partial t} = GH$$

for some $H \in \mathbb{K}[s,t]$. In this case, H is called the cofactor of G.

Definition 3.9. A differentiable function $W(s,t)$ on two variables s, t with coefficients in $\mathbb{K}$ is a first integral of the planar rational system

$$\begin{cases} s' = M(s,t) \\ t' = N(s,t) \end{cases} \tag{5}$$

where M, N are rational functions on s, t with coefficients in $\mathbb{K}$, if it is not a constant function and

$$M \frac{\partial G}{\partial s} + N \frac{\partial G}{\partial t} = 0$$

If furthermore W is a rational function, it is called a *rational first integral*.

It is not hard to see that the set of all first integrals of a planar rational system together with constant functions has an algebraic structure as a field. The intersection of such field and $\mathbb{K}(s,t)$ is the set of all rational first integrals with constants in $\mathbb{K}$. If the planar differential system has a rational first integrals, there is a non-composite reduced rational function, say F, such that every rational first integral has the form $u(F(s,t))$ for some univariate rational function u with coefficients in $\mathbb{K}$ (see [4]). In the other words, the set of all rational first integrals of the planar rational system is either an empty set or $\mathbb{K}(F) \setminus \mathbb{K}$, where $\mathbb{K}(F)$ is the field extended from $\mathbb{K}$ by F. Such the F is unique up to a composition with a homography. In particular, instead of finding all rational first integrals, looking for a non-composite one is enough.

On the other hand, the set of rational first integrals, and all invariant algebraic curves of a planar rational system does not change if we multiply the right hand side of the two differential equations of the system by the same non-zero rational function in $\mathbb{K}(s,t)$. Therefore it is suffices to consider planar polynomial systems for studying invariant algebraic curves and rational first integrals. Furthermore, by multiplying the right hand side of the differential equations in the system (3) by $\frac{M_2 N_2}{gcd(M_1 N_2, M_2 N_1)}$, one can always assume that M, N are polynomials such that $gcd(M,N) = 1$.

The following theorem is a classical result on relation between irreducible invariant algebraic curves and rational first integrals of a planar rational system. We recall here for technique purpose. For further detail, we prefer to many classical literatures about rational first integrals, for instance, see [26].

Theorem 3.10. *There is a natural number N such that a given planar rational system has a rational first integral if and only if the system has more than N irreducible invariant algebraic curves. Furthermore, if $W = \frac{P}{Q}$ is a reduced rational first integral then every irreducible invariant algebraic curves is defined by an irreducible factor of $c_1 P - c_2 Q$, where c_1, c_2 are arbitrary constants.*

Proposition 3.11. *If the parametriazable first-order AODE $F(x, y, y') = 0$ has an algebraic general solution, then its associated differential system with respect to a proper parametrization has a rational first integral.*

Proof. If the differential equation $F(x, y, y') = 0$ has an algebraic general solution, then so is its associated system. In this case, the associated system must have an irreducible invariant algebraic curve $G(s, t) = 0$ such that G is monic and at least one of the coefficients of G contains a constant which is transcendental over $\mathbb{K}$. In the other words, the associated system has infinitely many irreducible invariant algebraic curves. Thus it has a rational first integral. □

Theorem 3.12. *Assume that $W = \frac{P}{Q}$ is a reduced rational first integral of the planar rational system*

$$\begin{cases} s' = \dfrac{M_1(s,t)}{M_2(s,t)} \\ t' = \dfrac{N_1(s,t)}{N_2(s,t)} \end{cases}$$

where $M_1, M_2, N_1, N_2 \in \mathbb{K}[s, t]$, and that $(s(x), t(x))$ is an algebraic solution in which not both $s(x)$ and $t(x)$ are constants. Then $(s(x), t(x))$ is an algebraic general solution if and only if $W(s(x), t(x))$ is a constant which is transcendental over $\mathbb{K}$.

Proof. Assume that $(s(x), t(x))$ is an algebraic general solution of the planar rational system, then

$$\begin{aligned} W'(s(x), t(x)) &= s'(x)\frac{\partial W}{\partial s}(s(x), t(x)) + t'(x)\frac{\partial W}{\partial t}(s(x), t(x)) \\ &= \left(M\frac{\partial W}{\partial s} + N\frac{\partial W}{\partial t}\right)(s(x), t(x)) = 0 \end{aligned}$$

Therefore $W(s(x), t(x)) = c$ is an arbitrary constant. If $c \in \mathbb{K}$, then $P - cQ \in \mathbb{K}[s, t]$ has an irreducible factor in $\mathbb{K}[s, t]$ vanished at $(s(x), t(x))$. It can not happen. Hence $c \notin \mathbb{K}$. Since $\mathbb{K}$ is algebraically closed, c is transcendental over $\mathbb{K}$.

Conversely, assume that $(s(x), t(x))$ is a non-constant algebraic solution of the given planar rational system such that $W(s(x), t(x)) = c$, where c is a constant being transcendental over $\mathbb{K}$. Let G be an irreducible polynomial such that $G(s(x), t(x)) = 0$. Since $P - cQ$ is also vanished a long $(s(x), t(x))$, G must be an irreducible factor of $P - cQ$. As in [29, Ch. 3, Thm. 3.6], G has the form $A + \alpha B$ for some $A, B \in \mathbb{K}[s,t]$, $B \neq 0$, and $\alpha \in \overline{\mathbb{K}(c)}$ which is still transcendental over $\mathbb{K}$.

Now let $H \in \mathbb{K}(x)\{s,t\}$ be a differential polynomial such that $H(s(x), t(x)) = 0$. We denote $\tilde{H} := \text{prem}(H, \{\tilde{M}, \tilde{N}\})$ where $\tilde{M} := M_2 s' - M_1$ and $\tilde{N} := N_2 t' - N_1$. To finish the proof, we need to show that $\tilde{H} = 0$. It is clear that $\tilde{H} \in \mathbb{K}(x)\{s,t\}$ and satisfies $\tilde{H}(s(x), t(x)) = 0$. Let consider both $G = A + \alpha B$ and $\tilde{H}$ as polynomials in s, t with coefficient in $\mathbb{K}(\alpha, x)$. Then they are both vanished along $(s(x), t(x))$, and G is, again, irreducible. Thus $\tilde{H}$ must be divisible by G. It is only possible in the case $\tilde{H} = 0$, because α is transcendental not only on $\mathbb{K}$ but also on $\mathbb{K}(x)$. Hence $(s(x), t(x))$ is a general solution. □

The following corollary is an immediately consequence of the above theorem. It help us to split a planar rational system into two autonomous first-order AODEs, which lead us to the algorithm for determining explicit algebraic general solution of a planar rational system.

Corollary 3.13. *Assume that $W = \frac{P}{Q}$ is a reduced rational first integral of the system*

$$\begin{cases} s' = \dfrac{M_1(s,t)}{M_2(s,t)} \\ t' = \dfrac{N_1(s,t)}{N_2(s,t)} \end{cases}$$

where $M_1, M_2, N_1, N_2 \in \mathbb{K}[s,t]$ and that $(s(x), t(x))$ is an algebraic general solution. Then

i. $s(x)$ is an algebraic general solution over $\mathbb{K}(c)$ of the autonomous first-order AODE $F_1(s', s) = 0$, where

$$F_1 := Res_t(P - cQ, M_2 s' - M_1)$$

ii. $t(x)$ is an algebraic general solution over $\mathbb{K}(c)$ of the autonomous first-order AODE $F_2(s', s) = 0$, where

$$F_2 := Res_s(P - cQ, N_2 s' - N_1)$$

Fortunately the problem of finding algebraic general solutions of autonomous first-order AODEs is investigated. In [1], Aroca et. al. proposed a criteria to decide whether an autonomous first-order AODE having an algebraic general solution and compute such solution in affirmative case. Combining the previous theorem and the corollary, together with the result of Aroca et. al., an algorithm for computing explicit algebraic general solutions of planar rational systems with a given rational first integral will be proposed next. For determining a rational first integral, one can use the package *RationalFirstIntegrals* which have been implementing by A. Bostan et. al. [4].

Algorithm 1 Algebraic general solution of a planar rational system

Require: The planar rational system

$$\begin{cases} s' = \dfrac{M_1(s,t)}{M_2(s,t)} \\ t' = \dfrac{N_1(s,t)}{N_2(s,t)} \end{cases}$$

and $W = \frac{P}{Q}$ a reduced rational first integral.

Ensure: An algebraic general solution $(s(x), t(x))$.

If $M = 0$, then $s(x) = c$ and $t(x)$ is an algebraic general solution of $t' = N(c, t)$
If $N = 0$, then $t(x) = c$ and $s(x)$ is an algebraic general solution of $s' = M(s, c_1)$
Compute $F_1 := \mathrm{Res}_t(P - c_1 Q, M_2(s,t)s' - M_1(s,t))$
$\mathcal{S} :=$ the set of all irreducible factors of F_1 in $\overline{\mathbb{K}(c)}[s', s]$ containing s'
for all $H \in \mathcal{S}$ **do**
    If $H(s', s) = 0$ has no algebraic solution, then return "No algebraic general solution"
    $s(x) :=$ an algebraic solution of $H(s', s) = 0$
    $t(x) :=$ a solution of the equation $W(s(x), t) = c_1$
    If $s'(x) - M(s(x), t(x)) = t'(x) - N(s(x), t(x)) = 0$, then return "$(s(x+c_2), t(x+c_2))$"
end for
Return "No algebraic general solution"

Example 3.14. Consider the palanar rational system

$$\begin{cases} s' = t \\ t' = \dfrac{t^2}{2s} \end{cases} \tag{6}$$

By multiplying the right hand sides of the two differential equations of the system with $\frac{2s}{t}$, we obtain a new system which shares the same set of rational first integrals and invariant algebraic curves:

$$\begin{cases} s' = 2s \\ t' = t \end{cases} \tag{7}$$

Using the package *RationalFirstIntegrals* of A. Bostan et. al. (see [4]), we can evaluate a non-composite rational first integral of the last system, for instance $W = \frac{-64s}{100s-t^2}$. W is also a non-composite rational first integral of the system (6). Now we can use the algorithm 1 to find an algebraic general solution of the system (6). First we set

$$F_1(s, s') := \mathrm{Res}_t(s' - t, -64s - c_1(100s - t^2)) = (64 + 100c_1)s - c_1 s'$$

which is an irreducible polynomial in $\overline{k(c_1)}[s, s']$. Solving the differential equation $F_1(s, s') = 0$ (by using Aroca's et. al. algorithm, or just by integrating) yields an algebraic solution:

$$s(x) = \frac{1}{c_1}(16 + 25c_1)x^2$$

Next, we find $t(x)$ by solving the algebraic equation $W(s(x), t) = c_1$. It gives two candidates $\frac{2}{c_1}(16 + 25c_1)x$ and $-\frac{2}{c_1}(16 + 25c_1)x$. By substituting them to the system (6), we see that

$$(s(x), t(x)) := \left(\frac{1}{c_1}(16 + 25c_1)x^2, \frac{2}{c_1}(16 + 25c_1)x\right)$$

is an algebraic solution. Since the system is autonomous, $(s(x + c_2), t(x + c_2))$ is an algebraic general solution.

Example 3.15. Consider the planar rational system

$$\begin{cases} s' = \dfrac{t^2}{2} \\ t' = \dfrac{t^3}{2s^2 - 1} \end{cases} \tag{8}$$

A rational first integral, for instance $W = \frac{s^2-1}{t^4}$, can be found by a process similar to the one in the previous example. Let

$$F_1(s, r) := \mathrm{Res}_t\left(sr - t^2, s^2 - 1 - ct^4\right) = \left(cs^2r^2 - s^2 + 1\right)^2$$

By solving the autonomous differential equation $F(s, s') = 0$ we obtain an algebraic solution, for instance,

$$s(x) = \pm\sqrt{\frac{x^2}{c} + 1}$$

Next we find $t(x)$ by solving the algebraic equation $W(s(x), t) = c$. Therefore, $t(x) = \pm\sqrt{\frac{x}{c}}$. Finally,

$$\left(\pm\sqrt{\frac{(x+d)^2}{c} + 1}, \pm\sqrt{\frac{x+d}{c}}\right)$$

are algebraic general solutions of the given planar rational system, where c and d are arbitrary constants.

3.3 Algebraic general solutions with degree bound

In this section, we will combine previous results to study further the problem of finding an algebraic general solution of a surface parametrizable first-order AODEs. In particular, given a surface parametrizable first-order AODE and a positive integer n, we will present an algorithm for finding an algebraic general solution whose irreducible annihilating polynomial has total degree less than or equal to n.

Consider the surface parametrizable first-order AODE $F(x, y, y') = 0$ with a given proper parametrization

$$\mathcal{P}(s, t) := (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t))$$

for some $\chi_1, \chi_2, \chi_3 \in \mathbb{K}(s, t)$. Assume that $y = y(x) \in \overline{K(x)}$ is an algebraic general solution of the differential equation $F(x, y, y') = 0$, where the field K is extended from $\mathbb{K}$ by transcendence constants. Let $Y(x, y) \in K[x, y]$ be an irreducible annihilating polynomial of $y(x)$. We sometimes call $Y(x, y) = 0$ an algebraic general solution instead of $y(x)$.

We denote $\deg Y, \deg_x Y$ as the total degree of $Y(x, y)$ and the degree of x in Y, respectively.

Theorem 3.16. *With notation as above, let $(\sigma_1(x, y, z), \sigma_2(x, y, z)) := \mathcal{P}^{-1}(x, y, z)$ be the inverse map of $\mathcal{P}$. If the differential equation $F(x, y, y') = 0$ has an algebraic general solution $Y(x, y) = 0$ with $\deg Y \leq n$, then the associated system has a rational first integral whose total degree is less than or equal to*

$$m := n^3 \left(\deg_x \sigma_1 + \deg_y \sigma_1 + \deg_x \sigma_2 + \deg_y \sigma_2\right) + 2n^2 \left(\deg_z \sigma_1 + \deg_z \sigma_2\right).$$

Proof. Denote $s(x) := \sigma_1(x, y(x), y'(x))$ and $t(x) := \sigma_2(x, y(x), y'(x))$, then the pair $(s(x), t(x))$ is an algebraic general solution of the associated system. Let $G(s, t) \in K[s, t]$ be an irreducible polynomial such that $G(s(x), t(x)) = 0$. $G(s, t) = 0$ is in fact an irreducible invariant algebraic curve of the associated system with coefficients in K. We will first claim that $\deg G \le m$.

Denote

$$Q_1(x,y) := \sigma_1\left(x, y, -\frac{\frac{\partial}{\partial x}Y(x,y)}{\frac{\partial}{\partial y}Y(x,y)}\right)$$

and

$$Q_2(x,y) := \sigma_2\left(x, y, -\frac{\frac{\partial}{\partial x}Y(x,y)}{\frac{\partial}{\partial y}Y(x,y)}\right)$$

which are rational functions in $\mathbb{K}(x,y)$. Then $s(x) = Q_1(x, y(x))$ and $t(x) = Q_2(x, y(x))$. The degree of x and y on Q_1 and Q_2 can be estimated in terms of σ_1, σ_2 and Y as follow:

$$\deg_x Q_1 \le n.\deg_x \sigma_1 + \deg_z \sigma_1 \tag{9}$$

$$\deg_y Q_1 \le n.\deg_y \sigma_1 + \deg_z \sigma_1 \tag{10}$$

$$\deg_x Q_2 \le n.\deg_x \sigma_2 + \deg_z \sigma_2 \tag{11}$$

$$\deg_y Q_2 \le n.\deg_y \sigma_2 + \deg_z \sigma_2 \tag{12}$$

Now in order to get annihilating polynomials of $s(x), t(x)$, using the resultant is a fast way. In particular, the polynomials

$$R_1(s,x) := \mathrm{Res}_y(numer(Q_1) - s.denom(Q_1), Y(x,y))$$
$$R_2(s,x) := \mathrm{Res}_y(numer(Q_2) - t.denom(Q_2), Y(x,y))$$

are annihilating polynomials of $s(x)$ and $t(x)$ respectively, where $numer(Q_1)$ is the numerator of Q_1 and $denom(Q_1)$ the denominator one. Therefore $H(s,t) := \mathrm{Res}_x(R_1(s,x), R_2(t,x))$ is a polynomial in $K[s,t]$ satisfying $H(s(x), t(x)) = 0$. It implies that G must be divide H. From the definition of the resultant, one can determine immediately an upper bound for the total degree of H, and thus of G. In fact,

$$\deg_s H \le \deg_s R_1.\deg_x R_2 \le N^2(\deg_x Q_2 + \deg_y Q_2)$$

Equivalently, we also have

$$\deg_t H \le N^2(\deg_x Q_1 + \deg_y Q_1)$$

Combining with (9), (10),(11) and (12) yields $\deg G \le m$.

Moreover, since $(s(x), t(x))$ is an algebraic general solution, $G(s, t = 0)$ can be seen as the class of all irreducible invariant algebraic curves of the associated system. Therefore its degree bound is also a degree bound for the non-composite rational first integral. □

As an immediate consequence, the theorem leads us to the following algorithm for finding an algebraic general solution $Y(x,y) = 0$ with $\deg Y \le n$ of the differential equation $F(x, y, y') = 0$.

Algorithm 2 Algebraic general solution of a first-order AODE with degree bound

Require: Differential equation $F(x, y, y') = 0$ with a proper parametrization $\mathcal{P}$, and a positive integer n.

Ensure: An algebraic general solution $Y(x, y) = 0$ such that $\deg Y \leq n$.

1: $(\sigma_1, \sigma_2) := \mathcal{P}^{-1}$

2: Determine a degree bound for a rational first integral for the associated differential system

$$m := n^3 \left(\deg_x \sigma_1 + \deg_y \sigma_1 + \deg_x \sigma_2 + \deg_y \sigma_2 \right) + 2n^2 \left(\deg_z \sigma_1 + \deg_z \sigma_2 \right)$$

3: Determine the associated differential system $\{s' = M, t' = N\}$, where

$$M(s,t) := \frac{\chi_3(s,t) \frac{\partial}{\partial t} \chi_1(s,t) - \frac{\partial}{\partial t} \chi_2(s,t)}{\frac{\partial}{\partial s} \chi_1(s,t) \frac{\partial}{\partial t} \chi_2 1(s,t) - \frac{\partial}{\partial t} \chi_1(s,t) \frac{\partial}{\partial s} \chi_2(s,t)}$$

$$N(s,t) := \frac{\frac{\partial}{\partial s} \chi_2(s,t) - \chi_3(s,t) \frac{\partial}{\partial s} \chi_1(s,t)}{\frac{\partial}{\partial s} \chi_1(s,t) \frac{\partial}{\partial t} \chi_2 1(s,t) - \frac{\partial}{\partial t} \chi_1(s,t) \frac{\partial}{\partial s} \chi_2(s,t)}$$

4: If the associated differential system has no rational first integral of total degree at most m, then return "No algebraic general solution of total degree at most n". Otherwise, go to next step.

5: $W :=$ a rational first integral of degree at most m of the system, and solving the system by using the algorithm 1

6: If the system has no algebraic general solution, then return "No algebraic general solution of total degree at most n"

7: $(s(x), t(x)) :=$ an algebraic general solution of the system

8: Compute $y(x) := \chi_2 \left(s(2x - \chi_1(s(x), t(x))), t(2x - \chi_1(s(x), t(x))) \right)$

9: $Y(x, y) :=$ an irreducible annihilating polynomial of $y(x)$

10: If $\deg Y > n$, then return "No algebraic general solution of total order at most n"

11: Return "$Y(x, y) = 0$".

Example 3.17. Consider the differential equation

$$y'^3 - 4xyy' + 8y^2 = 0 \tag{13}$$

The solution surface is rational, because it admits the proper parametrization

$$\mathcal{P}(s,t) := \left(\frac{t^3 + 8s^2}{4st}, s, t \right)$$

The inverse map of the parametrization is $(\sigma_1(x,y,z), \sigma_2(x,y,z)) := (y,z)$. The associated system of the given differential equation with respect to $\mathcal{P}$ is

$$\begin{cases} s' = t \\ t' = \dfrac{t^2}{2s} \end{cases}$$

If we look for an algebraic general solution $Y(x,y) = 0$ with $\deg Y \leq 2$, then we need to find a rational first integral of total degree at most 16 of the associated system. As we have seen in previous example, the associated system has the rational first integral $W = \frac{64s}{100s - t^2}$ of total degree 2, and the algebraic general solution $(s(x), t(x)) := \left(\frac{1}{c_1}(16 + 25c_1)(x + c_2)^2, \frac{2}{c_1}(16 + 25c_1)(x + c_2) \right)$. By apply the theorem 3.5, we have

$$y(x) = \frac{1}{c_1^3}(c_1 x - 25c_1 - 16)(16c_1 x + 25c_1^2 x - 625c_1^2 - 800c_1 - 256)$$

is an algebraic general solution of the given differential equation.

4 Rational general solutions of first-order AODEs

This section is devoted for studying rational general solutions of first-order AODEs. A general solution contains an arbitrary constant. A rational general solution in which the constant appears rationally is called strong. In this section, we present a full algorithm for determining a strong rational general solution for a first-order AODE.

In order to obtain the algorithm, we also approach the differential equation from a geometric point of view. However, different from the previous section, we are going to view first-order AODEs as algebraic curves over the field of algebraic functions. We intrinsically use parametrization of algebraic curves to transform the differential equation to a first-order first-degree AODE (see Section 4.3). Parametrizations to be used must be "good" enough to make sure that every coefficient appears during the transformation is a rational function. In order to do that, we study some properties of optimal parametrizations for rational curves over the field of rational functions (see Section 4.2). Among first-order first-degree AODEs, only Riccati and linear differential equations potentially admit a rational general solution. This leads us to a decision algorithm for determining a strong rational general solution of a first-order AODE (see Section 4.4). This section is based on author's work in [34, 12].

4.1 Strong rational general solution

In this section, we give a necessary condition for a first-order AODE to admit a rational general solution of the form $y(x,c) \in \mathbb{K}(x,c) \setminus \mathbb{K}(x)$, where c is a transcendental constant. Consider a first-order AODE, $F(x,y,y')=0$, for an irreducible polynomial F. We view the equation to be an algebraic one by replacing the derivative by an independent variable, i. e. $F(x,y,z)=0$.

Definition 4.1. The algebraic curve $\mathcal{C}_F$ over $\overline{\mathbb{K}(x)}$ defined by $F(x,y,z)=0$ is called the *corresponding curve* of the differential equation $F(x,y,y')=0$.

The following theorem is a slightly different version of Theorem 2.4 in [6]. Note, that we assume irreducibility in $\mathbb{K}[x,y,z]$.

Theorem 4.2. *Let F be an irreducible polynomial in $\mathbb{K}[x,y,z] \setminus \mathbb{K}[x,y]$. If the differential equation $F(x,y,y')=0$ has a rational solution of the form $y(x,c) \in \mathbb{K}(x,c) \setminus \mathbb{K}(x)$ for an arbitrary constant c, then its corresponding curve in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$ is rational, and admits a parametrization with coefficients in $\mathbb{K}(x)$.*

Proof. First, we need to prove that F is still irreducible as a polynomial in $\overline{\mathbb{K}(x)}[y,z]$. In order to do that, let us consider the ideal

$$I := \{H \in \overline{\mathbb{K}(x)}[y,z] \mid H(x,y(x,c),y'(x,c))=0\}$$

in the polynomial ring $\overline{\mathbb{K}(x)}[y,z]$. We claim that I is a principle prime ideal. Consider the ring homomorphism $\phi : \overline{\mathbb{K}(x)}[y,z] \to \overline{\mathbb{K}(x)}(c)$, defined by $\phi(H) := H(x,y(x,c),y'(x,c))$ for $H \in \overline{\mathbb{K}(x)}[y,z]$. The kernel of ϕ is exactly I. Therefore ϕ induces an embedding from the quotient ring $\overline{\mathbb{K}(x)}[y,z]/I$ to $\overline{\mathbb{K}(x)}(c)$. Thus $\overline{\mathbb{K}(x)}[y,z]/I$ is a domain, and then I is a prime ideal. Since $\overline{\mathbb{K}(x)}[y,z]$ is a noetherian unique factorization domain, we know from [13, Prop. 1.12A, p. 7] that every prime ideal of height one is principle. Hence, I is principle.

Next we prove that I can be generated by an irreducible polynomial G in $\mathbb{K}[x,y,z]$. We construct such a generator by the method of Gröbner bases. Let $y(x,c) = \frac{P_1(x,c)}{P_2(x,c)}$ and $y'(x,c) = \frac{Q_1(x,c)}{Q_2(x,c)}$ be in reduced form, i. e. $P_1,P_2,Q_1,Q_2 \in \mathbb{K}[x,c]$ such that $\gcd(P_1,P_2) = \gcd(Q_1,Q_2) = 1$. From the definition of the ideal I, we know by implizitation that

$$I = \langle yP_2 - P_1,\ zQ_2 - Q_1, 1 - P_2t_1, 1 - Q_2t_2 \rangle \cap \overline{\mathbb{K}(x)}[y,z]\,.$$

In which the first component of the right hand side is an ideal in $\overline{\mathbb{K}(x)}[c,t_1,t_2,y,z]$ generated by the polynomials $yP_2 - P_1, zQ_2 - Q_1, 1 - P_2t_2$ and $1 - Q_2t_2$. We fix the lexicographic ordering on $\overline{\mathbb{K}(x)}[c,t_1,t_2,y,z]$ with $c > t_1 > t_2 > y > z$. Using this ordering we compute a reduced Gröbner basis of I by first computing a reduced Gröbner basis for the first component of the right hand side, and then eliminating all elements containing c, t_1, t_2. Buchberger's algorithm and reduction of the obtained basis yields a list of polynomials in the variables c, t_1, t_2, y, z with coefficients in $\mathbb{K}(x)$. Therefore, after eliminating polynomials containing c, t_1, t_2, we obtain a reduced Gröbner basis of I which contains only polynomials in $\mathbb{K}(x)[y, z]$. Since I is principle, the reduced Gröbner basis of I contains only one element, say $G_1 \in \mathbb{K}(x)[y, z]$. Moreover, since I is a prime ideal, G_1 must be irreducible over $\overline{\mathbb{K}(x)}[y, z]$ and hence also in $\mathbb{K}(x)[y, z]$. Let $G \in \mathbb{K}[x, y, z]$ such that $G_1 = \frac{a(x)}{b(x)} G$ for some $a(x), b(x) \in \mathbb{K}[x]$ and G is primitive over $\mathbb{K}[x]$. Hence, G is irreducible over $\mathbb{K}(x)[y, z]$ (since G_1 is irreducible). Then we have $I = \langle G_1 \rangle = \langle G \rangle$ over $\overline{\mathbb{K}(x)}[y, z]$. Therefore, G is irreducible over $\overline{\mathbb{K}(x)}[y, z]$.

Since F is an irreducible element in the ideal I, F differs from G by multiplication with a non-zero constant factor in $\mathbb{K}$. Therefore, F is also irreducible over $\overline{\mathbb{K}(x)}[y, z]$.

By now, the corresponding curve $\mathcal{C}_F$ is irreducible. Since $F(x, y(x, c), y'(x, c)) = 0$, $\mathcal{C}_F$ can be parametrized by a pair of rational functions $\mathcal{P}(t) := (y(x, t), \frac{\partial}{\partial x} y(x, t))$. Hence $\mathcal{C}$ is rational. □

Theorem 4.2 motivates the following definitions.

Definition 4.3. The first-order AODE, $F(x, y, y') = 0$, is called *parametrizable* if its corresponding curve is rational.

A parametrizable first-order AODE is surface parametrizable. But the converse direction is not always true. In fact, we will see in Section 4.2 that if a first-order AODE is parametrizable, then its corresponding curve can be parametrized by a pair $(p_1(x, t), p_2(x, t))$ of rational functions in x and t. In this case, $(x, p_1(x, t), p_2(x, t))$ is a rational parametrization for the corresponding surface. However, it is easy to check that the differential equation

$$y'^2 - y^3 - x = 0$$

is surface parametrizable but not parametrizable.

All differential equations of the form $y' F_1(x, y) = F_0(x, y)$, where $F_0, F_1 \in \mathbb{K}[x, y]$, are parametrizable. As a consequence, we might also say that all quasi-linear differential equations of the form $y' = \frac{F_0(x,y)}{F_1(x,y)}$ are parametrizable.

Note, that almost all of the first-order AODEs listed in the collection of Kamke [17] are parametrizable. In fact 89 percent are parametrizable AODEs. The remaining ones consist of two classes. One part contains the reducible AODEs, hence, parametrizability of the factors can be considered. Around one half of the reducible AODEs have parametrizable factors. The other part consists of AODEs or which the corresponding curve has genus greater than 0.

The class of first-order AODEs covers around 64 percent of the entire collection of first-order ODEs in Kamke. Some of the remaining ODEs contain arbitrary functions. For certain choices of these functions, the ODEs might be algebraic. For further details on statistical investigations of Kamke's list we refer to [12].

A rational general solution of a first-order AODE is not necessary of the form $y(x,c) \in \mathbb{K}(x,c) \setminus \mathbb{K}(x)$ for some transcendental constant c. However, if the $y(x,c)$ is a solution of a first-order AODE, then it is a general solution in the sense of Ritt. In fact, let assume that $H \in K(x)\{y\}$ be an arbitrary differential polynomial such that $H(y(x,c)) = 0$, and that $G := \text{prem}(H, F)$. Then $G \in \mathbb{K}(x)[y, y']$. From the definition of pseudo differential remainder, we know that there are natural numbers m, n such that $S_F^m I_F^n G - H$ is a linear combination of F and its derivatives with coefficients in $\mathbb{K}(x)\{y\}$, where S_F and I_F are separant and initial of F respectively. S_F and I_F are not vanished at $y = y(x,c)$. Otherwise, as we have seen in the proof of Theorem 4.2, that S_F and I_F are different from F by multiplying a rational function in $\mathbb{K}(x)$, which is not possible. Therefore G is vanished at $y = y(x,c)$. It implies that G is different from F by multiplying a rational function in $K(x)$. This implies $G = 0$. Hence $y(x,c)$ is a general solution.

Definition 4.4. A solution y of the differential equation $F(x, y, y') = 0$ is called a *strong rational general solution* if $y = y(x,c) \in \mathbb{K}(x,c)\setminus\mathbb{K}(x)$, where c is a transcendental constant over $\mathbb{K}(x)$.

Theorem 4.2 is not true if the given rational general solution is not strong. For instance, the differential equation

$$x^3 y'^3 - (3x^2 y - 1)y'^2 + 3xy^2 y' - y^3 + 1 = 0$$

has a rational general solution

$$y(x) = cx + (c^2 + 1)^{\frac{1}{3}},$$

which is not strong. The corresponding curve has genus 1. Therefore, the differential equation has no strong rational general solution. However, as we will see later, if a parametrizable first-order AODE has a rational general solution, then it has a strong rational general solution.

4.2 Optimal parametrization of rational curves

We have seen that the corresponding curve of a first-order AODE having a strong rational general solution is rational. Moreover, by Theorem 4.2 the corresponding curve admits a parametrization with coefficients in $\mathbb{K}(x)$. In case we have a parametrization with coefficients in $\mathbb{K}(x)$ we can decide the existence of a strong rational general solution and compute it. Indeed, as we show in this section, such a parametrizations always exists.

Optimal parametrization is a key notion to answer the question. Several algorithms for determining an optimal parametrization of a rational curve were provided. In [31], Sendra, Winkler and Pérez-Díaz proposed an algorithm for computing an optimal parametrization of a rational curve over the field $\mathbb{Q}$ of rational numbers. Similar result for the class of rational curves over the field $\mathbb{Q}(x)$ of rational functions is presented in [14]. From a different method, Beck and Schicho studied the optimal parametrization problem for rational curves over perfect fields [2]. Since $\mathbb{K}(x)$ is a perfect field, the algorithm of Beck and Schicho is applicable over $\mathbb{K}(x)$. Below, we follows the idea by Hilgarter and Winkler [14] to describe the field of an optimal parametrization of a rational curve over $\mathbb{K}(x)$.

Let us fix a rational curve $\mathcal{C}$ in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$ defined by $G(x,y,z)=0$ for some irreducible polynomial $G \in \mathbb{K}(x)[y,z]$. As a consequence of Hilbert-Hurwitz theorem [31, Ch. 5, p. 152], $\mathcal{C}$ can be rationally transformed down to a line or a conic over $\mathbb{K}(x)$, depending on whether the total degree of G is odd or even, respectively. The transformation was described in [31] by using the notion of adjoint curves. The line is always parametrizable over $\mathbb{K}(x)$. To parametrize the conic, it is sufficient to search for a $\mathbb{K}(x)$-rational point on it.

In the following we show, along the lines of [14, 31], that indeed there always exists such a $\mathbb{K}(x)$-rational point. Let us consider the projective conic $\mathcal{E} \in \mathbb{P}^2(\overline{\mathbb{K}(x)})$ defined by $G(y,z,w)=0$, where

$$G(y,z,w) := A_1y^2 + A_2yz + A_3z^2 + A_4yw + A_5zw + A_6w^2$$

is a polynomial in $\mathbb{K}[x][y,z,w]$ such that $(A_1,A_2,A_3) \neq (0,0,0)$. Our next goal is to determine a $\mathbb{K}(x)-$rational point of $\mathcal{E}$.

Without loss of generality, we may assume that $A_1 \neq 0$. Otherwise, we just swap y with z or w. Then G can be written as

$$\begin{aligned}G(y,z,w) = A_1\left(y + \frac{A_2}{2A_1}z + \frac{A_4}{2A_1}w\right)^2 + \left(\frac{4A_1A_3 - A_2^2}{4A_1}\right)z^2 + \\ + \left(\frac{2A_1A_5 - A_2A_4}{2A_1}\right)zw + \left(\frac{4A_1A_6 - A_4^2}{4A_1}\right)w^2\end{aligned}$$

If $4A_1A_3 - A_2^2 = 0$, we see immediately that $G\left(\frac{A_2}{2A_1}, -1, 0\right) = 0$. Therefore $\left(\frac{A_2}{2A_1}, -1, 0\right) \in \mathbb{P}^2(\mathbb{K}(x))$ is a $\mathbb{K}(x)-$rational point of $\mathcal{E}$. In general, if $4A_1A_3 - A_2^2 = 0$ or $4A_1A_6 - A_4^2 = 0$ or $4A_3A_6 - A_5^2 = 0$, the conic $\mathcal{E}$ is called a parabola. However, the condition for a conic to be a parabola does not invariant under linear projective transformations. In other words, a parabola can be transformed to a conic which is not a parabola by using a suitable linear projective map.

Let us assume that $4A_1A_3 - A_2^2 \neq 0$. We rewrite G as follow:

$$G(y,z,w) = \overline{A}_1\left(y + \frac{A_2}{2A_1}z + \frac{A_4}{2A_1}w\right)^2 + \overline{A}_2\left(z + \frac{2A_1A_5 - A_2A_4}{4A_1A_3 - A_2^2} \cdot w\right)^2 + \overline{A}_3w^2$$

where

$$\overline{A}_1 = A_1$$
$$\overline{A}_2 = \frac{4A_1A_3 - A_2^2}{4A_1}$$
$$\overline{A}_3 = \frac{4A_1A_6 - A_4^2}{4A_1} - \frac{(2A_1A_5 - A_2A_4)^2}{4A_1\left(4A_1A_3 - A_2^2\right)}$$

Therefore, by using the linear transformation

$$\begin{bmatrix} \overline{y} \\ \overline{z} \\ \overline{w} \end{bmatrix} = \begin{bmatrix} 1 & \frac{A_2}{A_1} & \frac{A_4}{A_1} \\ 0 & 1 & \frac{2A_1A_5 - A_2A_4}{4A_1A_3 - A_2^2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \\ w \end{bmatrix}$$

the conic $\mathcal{E}$ can be transformed to a projective conic which is defined by

$$\overline{A}_1\overline{y}^2 + \overline{A}_2\overline{z}^2 + \overline{A}_3\overline{w}^2 = 0$$

Moreover, by multiplying both side of the equation by the common denominator, we may assume that $\overline{A}_1, \overline{A}_2, \overline{A}_3$ are polynomials.

Next, let $\overline{A}_1\overline{A}_3 = AP^2$ and $\overline{A}_2\overline{A}_3 = BQ^2$ for some $A, B, P, Q \in \mathbb{K}[x]$ such that A and B are square-free polynomials. We transform the previous conic one more time by using the following linear transformation:

$$\begin{bmatrix} Y \\ Z \\ W \end{bmatrix} = \begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & \overline{A}_3\sqrt{-1} \end{bmatrix} \cdot \begin{bmatrix} \overline{y} \\ \overline{z} \\ \overline{w} \end{bmatrix}$$

The obtained conic is the one defined by $AY^2 + BZ^2 - W^2 = 0$. By abuse of notation, we rename this conic by $\mathcal{E}$. Note that, the above transformations are bijective if $\overline{A}_3 \neq 0$ and easy to computer the inverse maps. (The case when $\overline{A}_3 = 0$ is trivial.)

Proposition 4.5. *For every square-free polynomials $A, B \in \mathbb{K}[x]$, the projective conic defined by $AY^2 + BZ^2 - W^2 = 0$ always has a $\mathbb{K}(x)-$rational point.*

Before giving a proof for this proposition, we need the following lemma.

Lemma 4.6. *Let A, B be polynomials in $\mathbb{K}[x]$ such that A is square-free and* $\deg A \geq \deg B \geq 1$. *Then there exists $a, b, m \in \mathbb{K}[x]$ such that a is square-free,* $\deg a < \deg A$, *and* $b^2 - B = am^2A$.

Proof. Denote by n the degree of A and let $x_1, \dots, x_n \in \mathbb{K}$ be roots of A. There exists a polynomial $b \in \mathbb{K}[x]$ of degree at most $n-1$ such that $b(x_i) = \sqrt{B(x_i)}$ for every $i = 1, \dots, n$, where $\sqrt{B(x_i)}$ is a square root of $B(x_i)$. We see that $B(x) \equiv b(x)^2 \mod (x - x_i)$ for every

$i = 1, \dots, n$. Since A is square-free, Chinese Remainder Theorem yields $B(x) \equiv b(x)^2 \mod A(x)$.

Now let $a, m \in \mathbb{K}[x]$ such that a is square-free and $\frac{b^2-B}{A} = a.m^2$. Note that such a pair (a, m) is always exist. It remains to prove that $\deg a < \deg A$. Indeed, we have

$$\begin{aligned}\deg a &= \deg(b^2 - B) - \deg(Am^2) \\ &\le \deg(b^2 - B) - \deg A \\ &\le \max\{2(\deg A - 1), \deg B\} - \deg A \\ &< \deg A\,.\end{aligned}$$

□

From the proof, we see that $\deg b \le \deg A - 1$. This fact leads us to an algorithmic way to determine the triple (a, b, m) by using indeterminate coefficient method. In particular, we first set b a polynomial of degree $\deg A - 1$ in x with indeterminate coefficients. The remainder of the division $b^2 - B$ by A can be computed by using Euclidean algorithm. Since A divides $b^2 - B$, the remainder must be equal to zero. This yields an algebraic system on the indeterminate coefficients. By solving the obtained algebraic system, we can find all possible choices for b, and hence for a and m.

Proof of Proposition 4.5. This proof follows the lines of [14].

Let $A, B \in \mathbb{K}[x]$ be square-free polynomials, and consider the projective conic $\mathcal{E}$ defined by $AY^2 + BZ^2 - W^2 = 0$. Denote $d(\mathcal{E}) := \min(\deg A, \deg B)$. We prove the existence of a $\mathbb{K}(x)-$rational point on $\mathcal{E}$ by induction on $d(\mathcal{E})$. In the induction base case, i. e. $d(\mathcal{E}) = 0$, for instance $\deg A = 0$, then $(1 : 0 : \sqrt{A}) \in \mathbb{P}^2(\mathbb{K}(x))$ is a $\mathbb{K}(x)-$rational point of the conic.

Let $m \ge 1$ be an arbitrary natural number, and assume that for every projective conic $\tilde{\mathcal{E}}$ defined by $\tilde{A}Y^2 + \tilde{B}Z^2 - W^2 = 0$ for some square-free polynomials $\tilde{A}, \tilde{B} \in \mathbb{K}[x]$, if $d(\tilde{\mathcal{E}}) < m$ then $\tilde{\mathcal{E}}$ admits a $\mathbb{K}(x)-$rational point. We need to prove that if $d(\mathcal{E}) = m$, then $\mathcal{E}$ also admits a $\mathbb{K}(x)-$rational point.

In case $d(\mathcal{E}) = m$, we process as follows. We may assume further that $\deg A \ge \deg B = m$, otherwise we just swap Y and Z. By Lemma 4.6, there exists $A_1, b, m \in \mathbb{K}[x]$ such that A_1 is square-free, $\deg A_1 < \deg A$, and $b^2 - B = A_1 m^2 A$. We transform the coordinate system (Y, Z, W) to the new one $(\overline{Y}, \overline{Z}, \overline{W})$ by the linear transformation

$$\begin{bmatrix} \overline{Y} \\ \overline{Z} \\ \overline{W} \end{bmatrix} = \begin{bmatrix} Am & 0 & 0 \\ 0 & b & 1 \\ 0 & B & b \end{bmatrix} \begin{bmatrix} Y \\ Z \\ W \end{bmatrix}.$$

Then we see that

$$A_1\overline{Y}^2 + B\overline{Z}^2 - \overline{W}^2 = (b^2 - B)(AY^2 + BZ^2 - W^2)\,.$$

Since B is square-free, $b^2 - B \neq 0$. Thus the conic $\mathcal{E}$ has a $\mathbb{K}(x)-$rational point if and only if the projective conic $\mathcal{E}_1$ defined by $A_1\overline{Y}^2 + B\overline{Z}^2 - \overline{W}^2 = 0$ has a $\mathbb{K}(x)-$rational point.

If $\deg A_1 < \deg B$, then $d(\mathcal{E}_1) = \deg A_1 < \deg B = m$. Therefore $\mathcal{E}_1$ satisfies the induction hypothesis. This show that $\mathcal{E}_1$ admits a $\mathbb{K}(x)-$rational point. Then so is $\mathcal{E}$.

In case $\deg A_1 \geq \deg B$, we can repeat the above process recursively until we get a projective conic $\mathcal{E}_k$ defined by $A_kY^2 + BZ^2 - W^2 = 0$, where A_k is square-free and $\deg A_k < \deg B$. Note that, the polynomial B does not change via these transformations. At this step we have $d(\mathcal{E}_k) = \deg A_k < \deg B = m$. In other words, $\mathcal{E}_k$ satisfies the induction hypothesis. Therefore $\mathcal{E}_k$ contains a $\mathbb{K}(x)-$rational point. Then so is $\mathcal{E}$. □

The proof is constructive. We now conclude the above discussion by the following theorem.

Theorem 4.7. *A rational curve defined over $\mathbb{K}(x)$, i. e. a curve which can be parametrized over $\overline{\mathbb{K}(x)}$, can actually be parametrized over $\mathbb{K}(x)$. Therefore optimal parametrizations of a rational curve over $\mathbb{K}(x)$ always have coefficients in $\mathbb{K}(x)$.*

Furthermore, an algorithm for determining such an optimal parametrization can be provided by following the process of Hillgarter and Winkler [14]. We summarize the discussion by a short description for the algorithm.

Algorithm 3 OPTIMALPARA (Optimal Parametrization)

Require: A rational curve $\mathcal{C}$ over $\mathbb{K}(x)$

Ensure: An optimal parametrization for $\mathcal{C}$

1: Determine a birational transformation, say $\mathcal{G}$, to transform the curve down to a conic, say $\mathcal{E}$, or a line by algorithm derived from the theorem of Hilbert-Hurwitz (see Theorem 5.8 and Algorithm HLBERT-HURWITZ in [31]). If it is a line, go to step 2. Otherwise, go to step 3.

2: Determine an optimal parametrization for the line, say $\mathcal{P}(t)$, and then return $\mathcal{G}^{-1}(\mathcal{P}(t))$.

3: Linearly transform the conic $\mathcal{E}$ to a projective conic of the form $AY^2 + BZ^2 - W^2 = 0$ for some $A, B \in \mathbb{K}[x]$ square-free polynomials.

4: Construct a $\mathbb{K}(x)-$rational point for the latter conic as the method described in this section.

5: Determine the corresponding $\mathbb{K}(x)-$rational point point in $\mathcal{E}$, say M.

6: Determine a parametrization, say $\mathcal{P}(t)$, for $\mathcal{E}$ by using the point M (see Algorithm CONIC-PARAMETRIZATION [31]).

7: Return $\mathcal{G}^{-1}(\mathcal{P}(t))$

4.3 Associated differential equation

In this section, we only work with the class of parametrizable first-order AODEs. Based on optimal parametrizations of the corresponding curves, we construct for each parametrizable first-order AODE an associated differential equation, which is a quasi-linear ordinary differential equation. Several facts about connections between rational general solutions of a parametrizable first-order AODE and its associated differential equation will be presented. The problem which remains is looking for rational general solutions of quasi-linear differential equations. This problem is discussed at the end of this section.

Consider a parametrizable first-order AODE $F(x, y, y') = 0$ and assume that an optimal parametrization $\mathcal{P} = p_1, p_2 \in (\mathbb{K}(x)(t))^2$ of the corresponding curve is given, where we write $p_i(t) = p_i(x, t)$ to indicate the dependence on x. Let $y(x) \in \overline{\mathbb{K}(x)}$ be an algebraic solution. Then the pair of two algebraic functions $(y(x), y'(x))$ can be seen as an algebraic solution point on the corresponding curve $\mathcal{C}$. Two cases arise.

(i) $(y(x), y'(x)) \notin \operatorname{im}(\mathcal{P})$, where $\operatorname{im}(\mathcal{P})$ is the image of $\mathcal{P}$. Then $(y(x), y'(x))$ can be determined from the finite set $\mathcal{C} \setminus \operatorname{im}(\mathcal{P})$.

(ii) $(y(x), y'(x)) = \mathcal{P}(\omega(x))$ for some $\omega(x) \in \overline{\mathbb{K}(x)}$. In this case we identify the algebraic function $\omega(x)$ with a point on the affine line $\mathbb{A}^1(\overline{\mathbb{K}(x)})$.

Let us take a look at the algebraic function $\omega(x)$. It satisfies the system

$$\begin{cases} p_1(x, \omega(x)) = y(x)\,, \\ p_2(x, \omega(x)) = y'(x)\,. \end{cases}$$

Therefore,

$$\frac{d}{dx} p_1(x, \omega(x)) = p_2(x, \omega(x))\,.$$

By expanding the left hand side, we have

$$\omega'(x) \cdot \frac{\partial p_1}{\partial t}(x, \omega(x)) + \frac{\partial p_1}{\partial x}(x, \omega(x)) = p_2(x, \omega(x))$$

Thus $\omega(x)$ either satisfies the algebraic relation

$$\begin{cases} \dfrac{\partial p_1}{\partial t}(x, \omega(x)) = 0\,, \\ \dfrac{\partial p_1}{\partial x}(x, \omega(x)) = p_2(x, \omega(x))\,, \end{cases}$$

or it is an algebraic solution of the quasi-linear differential equation

$$\omega' = \frac{p_2(x, \omega) - \frac{\partial p_1}{\partial x}(x, \omega)}{\frac{\partial p_1}{\partial t}(x, \omega)}\,. \tag{14}$$

The ODE in (14) will be of further importance.

Definition 4.8. Let $F(x,y,y')=0$ be an AODE and let $\mathcal{P}(t)=(p_q(x,t),p_2(x,t))$ be a proper rational parametrization of the corresponding curve. Then the ODE (14) is called the *associated differential equation.*

In the above, we have proven the following lemma.

Lemma 4.9. *With notations as above, if $y=y(x)\in\overline{\mathbb{K}(x)}$ is an algebraic solution of the differential equation $F(x,y,y')=0$, then one of the following holds:*

(i) The algebraic solution point $(y(x),y'(x))$ lies in the finite set $\mathcal{C}\setminus\mathrm{im}(\mathcal{P})$.

(ii) $y(x)=p_1(x,\omega(x))$ for some algebraic solution $\omega(x)$ of the algebraic system:

$$\begin{cases} \dfrac{\partial p_1}{\partial t}(x,\omega)=0\,, \\ \dfrac{\partial p_1}{\partial x}(x,\omega)=p_2(x,\omega)\,. \end{cases}$$

(iii) $y(x)=p_1(x,\omega(x))$ for some algebraic solution $\omega(x)$ of the associated quasi-linear differential equation (14).

Theorem 4.10. *We use the notation from above and assume that the parametrization $\mathcal{P}$ is proper. Then there is a one-to-one correspondence between rational general solutions of the differential equation $F(x,y,y')=0$ and rational general solutions of its associated differential equation (14).*

In particular, if $\omega(x)$ is a rational general solution of the associated equation (14), then $y(x)=p_1(x,\omega(x))$ is a rational general solution of given differential equation.

Conversely, if $y(x)$ is a rational general solution of the given differential equation, then $\omega(x)=\mathcal{P}^{-1}(y(x),y'(x))$ is a rational general solution of the associated equation (14), where $\mathcal{P}^{-1}$ is a rational representation of the inverse of $\mathcal{P}$.

Proof. Assume that $\omega(x)$ is a rational general solution of the associated differential equation (14), and denote $y(x):=p_1(x,\omega(x))$. From the construction above, it is clear that $y(x)$ is a rational solution of the differential equation $F(x,y,y')=0$.

It remains to show that $y(x)$ is a general solution. Let $G\in\mathbb{K}(x)\{y\}$ be a differential polynomial such that $G(y(x))=0$, and let $H:=\mathrm{prem}(G,F)$. We need to show that $H=0$. Since y' is the highest derivative occurring in F, we know that $H\in\mathbb{K}(x)[y,y']$. Both G and F vanish at $y(x)$, hence so does H regarded as a differential polynomial. Therefore, $H(\mathcal{P}(\omega(x)))=H(y(x),y'(x))=0$ regarding H as a polynomial. Note, that $(H\circ\mathcal{P})(\omega)=H(f_1(x,\omega),f_2(x,\omega))\in k(x,\omega)$. In order to fulfill $(H\circ\mathcal{P})(\omega)=0$, ω has to be in $\overline{\mathbb{K}(x)}$. Since $\omega(x)$ is a general solution of the associated differential equation, it contains an arbitrary constant and hence, $H\circ\mathcal{P}=0$. Therefore, $H=(H\circ\mathcal{P})\circ\mathcal{P}^{-1}=0$.

Equivalently, if $y(x)$ is a rational general solution of the given differential equation, then, by the construction of the associated equation, $\omega(x):=\mathcal{P}^{-1}(y(x),y'(x))$ is a rational solution of (14). By a similar argument as above ω is a rational general solution of the associated differential equation (14). □

Lemma 4.9 tells us that for finding rational solutions of a parametrizable first-order AODE, working with the class of quasi-linear first-order ODEs is essentially enough. If we look for rational general solutions, the situation is even much stricter. In fact, in [3], Behloul and Cheng proved that if a quasi-linear differential equation has infinitely many rational solutions, then it must be either a linear differential equation or a Riccati equation. The following theorem is a combination of Theorem 4.10 and the result of Behloul and Cheng.

Theorem 4.11. *Let $F(x, y, y') = 0$ be a first-order AODE.*

(i) If $F = 0$ has a strong rational general solution, then it is parametrizable and its associated differential equation is of the form

$$\omega' = a_0(x) + a_1(x)\omega + a_2(x)\omega^2 \,, \tag{15}$$

for some $a_0, a_1, a_2 \in \mathbb{K}(x)$.

(ii) If $F = 0$ is parametrizable and has a rational general solution, then its associated quasi-linear differential equation is of the form (15).

Proof. If a parametrizable first-order AODE has a rational general solution, then so does its associated differential equation. In this case the associated differential equation has infinitely many rational solutions. Then (ii) follows from the result of Behloul and Cheng in [3]. Finally, (i) follows immediately from Theorem 4.2 and (ii). □

Corollary 4.12. *If a parametrizable first-order AODE has a rational general solution, then it has a strong rational general solution.*

Proof. It is a consequence of the previous theorem and [30, Cor. 2.1, p. 18] □

We are looking for rational general solutions of first-order AODEs. The problem remained now is computing a rational general solution of the differential equation (15). In the case $a_2 = 0$, it is a linear differential equation of degree 1 which can be easily solved by integrating. In the case $a_2 \neq 0$, it is a classical Riccati equation.

For the problem of computing rational general solution, or even all rational solutions, of a Riccati equation, readers can refer [19] for a completed algorithm. In [19], Kovacic proposes an algorithm for computing Liouvilian solutions of a linear second order ODE. As a special case, Section 3.1 in that paper leads to a full algorithm for determining all rational solutions of a Riccati equation. Note that for a Riccati equation, the notion of rational general solutions and strong rational general solutions are coincide. In [6], Chen and Ma do a slight modification of the algorithm by Kovacic to seek for only strong rational general solution.

4.4 Algorithm and Examples

This section is devoted to an algorithm for finding strong rational general solutions of first-order AODEs. As we have seen before, if a first-order AODE has a strong rational general solution, then it is parametrizable, i. e. its corresponding curve is rational. Whenever a first-order AODE is parametrizable, the notions of rational general solution and strong rational general solution coincide. Moreover, in the case of having a strong rational general solution, the associated ODE is either a linear differential equation or a Riccati equation.

In Algorithm 4 we present a full algorithm which computes for a given first-order AODE a strong rational general solution, if it exists. Otherwise it decides that such a solution cannot exist.

Algorithm 4 Strong rational general solutions of first-order AODEs

Require: A first-order AODE, $F(x, y, y') = 0$, where $F \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$ is irreducible.
Ensure: A strong rational general solution $y(x)$, or "No strong rational general solution exists".

if genus of the corresponding curve is zero **then**
    Use Algorithm 3 to compute an optimal parametrization of the corresponding curve, say $(p_1(x,t), p_2(x,t)) \in (\mathbb{K}(x)(t))^2$.
    Compute

$$f(x,t) := \frac{p_2(x,t) - \frac{\partial}{\partial x} p_1(x,t)}{\frac{\partial}{\partial t} p_1(x,t)}.$$

    if $f(x,t)$ has the form $a_0(x) + a_1(x)t + a_2(x)t^2$ for some $a_0, a_1, a_2 \in \mathbb{K}(x)$ **then**
        Computing a rational general solution of the linear or Riccati equation $\omega' = f(x, \omega)$.
        if $\omega = \omega(x)$ is a rational general solution **then**
            return $y(x) = p_1(x, \omega(x))$
        end if
    end if
end if
return "No strong rational general solution exists".

Theorem 4.13. *Algorithm 4 returns a strong rational general solution of the given first-order AODE, $F(x, y, y') = 0$, if there is any, and it returns "No strong rational general solution exists" if the differential equation has no strong rational general solution.*

Hence, Algorithm 4 decides the existence of strong rational general solutions of the whole class of first-order AODEs. Furthermore, due to Corollary 4.12, Algorithm 4 can also be used for determining the existence of rational general solutions of parametrizable first-order AODEs. In the affirmative case it always computes such a solution.

Example 4.14 (Example 1.537 in Kamke [17]). Consider the differential equation

$$\begin{aligned}F(x,y,y') &= x^3y'^3 - 3x^2yy'^2 + (x^6+3xy^2)y' - y^3 - 2x^5y \\ &= (xy'-y)^3 + x^6y' - 2x^5y = 0\,.\end{aligned}$$

The associated curve defined by $F(x,y,z)=0$ can be parametrized by

$$\mathcal{P}(t) = \left(-\frac{t^3x^5 - t^2x^6 + (t-x)^3}{t^3x^5}, -\frac{2t^3x^5 - 2t^2x^6 + (t-x)^3}{t^3x^6}\right)\,.$$

Therefore, the associated differential equation with respect to $\mathcal{P}$ is

$$\omega' = \frac{1}{x^2}\cdot\omega\cdot(2\omega - x)\,,$$

which is a Riccati equation. By applying the algorithm by Kovacic, we can determine a rational general solution of the last differential equation, such as $\omega(x) = \frac{x}{1+cx^2}$. Hence, the differential equation $F(x,y,y')=0$ has the rational general solution $y(x) = cx(x+c^2)$.

Observe, that this is just an arbitrary example from the collection of Kamke [17]. In total around 64 percent of the listed ODEs there are AODEs and almost all of them are parametrizable and hence suitable for Algorithm 4. For further detail see [12].

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