Algorithmic Aspects of Graph Connectivity

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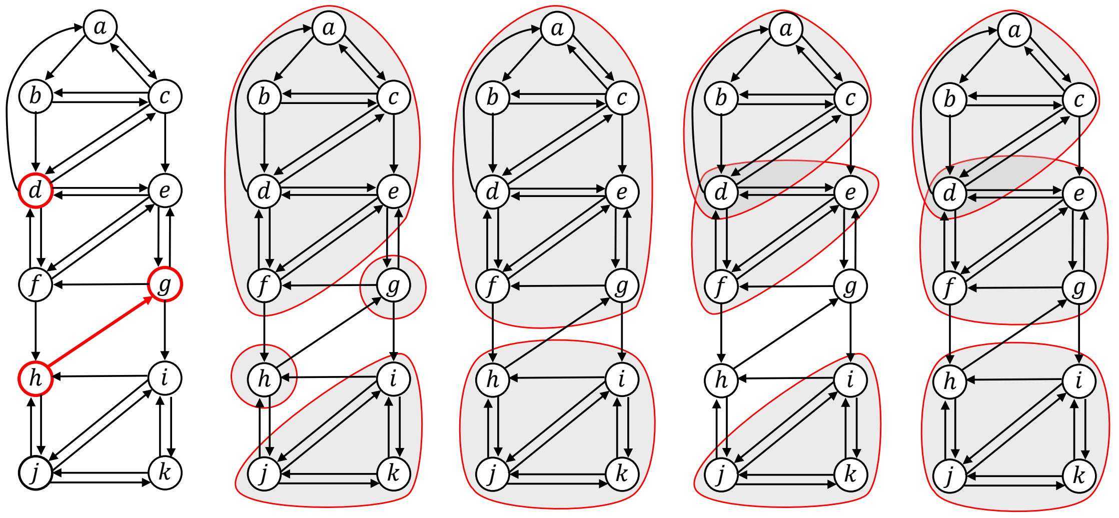
Connectivity is one of the most well-studied notions in graph theory. The literature covers many different aspects of connectivity related problems. During my research stay at the University of Michigan, we studied three different aspects of graph connectivity: (1) Connectivity oracles for graphs subject to vertex failures, (2) the characterization of vertex cuts with generalizations of SPQR-trees, and (3) faster algorithms for maximal k-connected induced subgraphs in directed graphs. For (1) the main goal was to improve upon the already almost tight results of [4]. For (2) the main goal was to re-evaluate the definitions of [3] to obtain a formal and clear expositions of their results as a basis for potential algorithmic uses of their graph decomposition. For both (1) and (2) related work was studied and several approaches were explored but no results were obtained. We plan to continue with (2) in future work. Therefore this report presents the results for (3), where faster algorithms were obtained. These results are joint work with Shiri Chechik, Thomas D. Hansen, Giuseppe F. Italiano, and Nikos Parotsidis, see [2].

1 Introduction

Problem definition and related concepts. *Strong connectivity.* Let $G = (V, E)$ be a directed graph (digraph) with $m = |E|$ edges and $n = |V|$ vertices. The digraph G is said to be *strongly connected* if there is a directed path from each vertex to every other vertex. The *strongly connected components* (SCCs) of G are its maximal strongly connected subgraphs. Two vertices $u, v \in V$ are *strongly connected* if they belong to the same strongly connected component of G.

2-edge connectivity. An edge of G is a *strong bridge* if its removal increases the number of strongly connected components. Let G be a strongly connected graph. We say that G is 2-edge-connected if it has no strong bridges. Two vertices v and w are 2-edge-connected if there are two edge-disjoint paths from v to w and two edge-disjoint paths from w to v. A 2-edge-connected component of G is a maximal subset of vertices such that any pair of distinct vertices is 2-edge-connected. For a set of vertices $C \subseteq V$ its induced subgraph $G[C]$ is a *maximal* 2*-edge-connected subgraph* of G if $G[C]$ is a 2-edge-connected graph and no superset of C has this property. The 2-edge-connected components of G might be very different from the maximal 2-edge-connected subgraphs of G because the two



(a) G (b) $2ECS(G)$ (c) $2ECC(G)$ (d) $2VCS(G)$ (e) $2VCC(G)$

Figure 1: (a) A strongly connected digraph G; strong articulation points and strong bridges are shown in red. (b) The 2-edge-connected subgraphs of G. (c) The 2-edge-connected components of G. (d) The 2-vertex-connected subgraphs of G. (e) The 2-vertex-connected components of G.

edge-disjoint paths between a pair of vertices of a 2-edge-connected component might use vertices that are not in the 2-edge-connected component. (See Figure 1 for an example.)

2-vertex connectivity. Analogous definitions can be given for 2-vertex connectivity. In particular, a vertex is a *strong articulation point* if its removal increases the number of strongly connected components of G. Let G be a strongly connected graph. The graph G is 2-vertex-connected if it has at least three vertices and no strong articulation points. Note that the condition on the minimum number of vertices disallows for degenerate 2-vertex-connected graphs consisting of two mutually adjacent vertices (i.e., two vertices v and w and the two edges (v, w) and (w, v)). Two vertices v and w are 2-vertex-connected if there are two internally vertex-disjoint paths from v to w and two internally vertex-disjoint paths from w to v, i.e., the paths meet at v and w but not in-between. A 2-vertex-connected component of G is a maximal subset of vertices such that any distinct pair of vertices is 2-vertex-connected. For a set of vertices $C \subseteq V$ its induced subgraph $G[C]$ is a *maximal 2-vertex-connected subgraph* of G if $G[C]$ is a 2-vertex-connected graph and no superset of C has this property. Note that the 2-vertex-connected components of G might be very different from the maximal 2-vertex-connected subgraphs of G.

k-connectivity. The notions of 2-edge and 2-vertex connectivity extend naturally to k-edge and k-vertex connectivity. Given a directed graph $G = (V, E)$, a set of edges S is an *edge cut* of size $|S|$ if its removal increases the number of strongly connected components of G. A strongly connected graph is k-edge-connected if it has no edge

cut of size less than k. Two vertices v and w are k-edge-connected if there are k edge-disjoint paths from v to w and k edge-disjoint paths from w to v. A k-edge-connected component of G is a maximal subset of vertices such that any pair of distinct vertices is k-edge-connected. For a set of vertices $C \subseteq V$ its induced subgraph $G[C]$ is a *maximal k-edge-connected subgraph* of G if $G[C]$ is a k-edge-connected graph and no superset of C has this property. A set of vertices S is a *vertex cut* of size $|S|$ if its removal increases the number of strongly connected components of G. A strongly connected graph G is k-vertex-connected if it has at least $k+1$ vertices and no vertex cut of size less than k. Two vertices v and w are k-vertex-connected if there are k internally vertex-disjoint paths from v to w and k internally vertex-disjoint paths from w to v. A k-vertex-connected component of G is a maximal subset of vertices such that any distinct pair of vertices is k-vertex-connected. For a set of vertices $C \subseteq V$ its induced subgraph $G[C]$ is a *maximal k-vertex-connected subgraph* of G if $G[C]$ is a k-vertex-connected graph and no superset of C has this property.

We usually omit the word *maximal* when referring to maximal k-edge- or k-vertex-connected subgraphs.

Our results. We present $O(m^{3/2})$ time algorithms for computing the maximal 2-edge-connected subgraphs and the maximal 2-vertex-connected subgraphs of a given directed graph with m edges and n vertices. This is an improvement over the existing $O(n^2)$ time algorithms [17] whenever m is $o(n^{4/3})$. The algorithm for 2-edge-connected subgraphs is extended to compute the maximal k-edge-connected subgraphs for any constant $k \geq 2$ and runs in time $O(m^{3/2} \log n)$, improving over the existing $O(n^2 \log n)$ time algorithm [17]. The maximal k-edge-connected (and k-vertex-connected) subgraphs are defined for undirected graphs as they are for directed graphs. We also show how to adjust the algorithm to compute the maximal k-edge-connected subgraphs for undirected graphs in time $O((m + n \log n)\sqrt{n})$, where k is again viewed as a constant. For the special case where $k = 3$, the running time for computing the 3-edge-connected subgraphs on undirected graphs is $O(m\sqrt{n})$.

Related work. In the literature the terms "components" and "blocks" have both been used to mean either k-connected components, as defined above, or the maximal (induced) k-connected subgraphs; therefore we explicitly use the term subgraphs for the latter in order to avoid further confusion.

Undirected graphs. It has been known for over 40 years how to compute the 2-edge- and 2-vertex-connected components of undirected graphs in linear time [28]. While the 2-edge-connected (resp., 2-vertex-connected) components are equal to the 2-edge-connected (resp., 2-vertex-connected) subgraphs in undirected graphs, this is no longer the case for $k > 2$. The first algorithm for computing the 3-vertex-connected components in linear (in the number of edges) time was by Hopcroft and Tarjan [19]. Later, Galil and Italiano [11] reduced the computation of the 3-edge-connected components to 3-vertex-connected components, thus obtaining a linear time algorithm for this case as well. Kanevsky and Ramachandran [22] showed how to test whether a graph is 4-vertex-connected in $O(n^2)$ time. Over 20 years ago, Nagamochi and Watanabe [27] presented an algorithm for computing the k-edge-connected components for $k > 3$ in $O(m + k^2 n^2)$ time. The best known algorithm for this problem runs in expected $\widetilde{O}(m + nk^3)$ time and was presented by Hariharan et al. [16]. Their algorithm additionally computes a partial version of the Gomory-Hu tree [15], that represents the edge-connectivity of the pairs whose edge-connectivity is less than k; the k-edge-connected components are contracted into singleton vertices in the tree. Karger [23] showed how to determine with high probability whether an undirected graph is k-edge-connected in $\widetilde{O}(m)$ time. In a recent breakthrough, Kawarabayashi and Thorup [24] presented a deterministic algorithm that achieves similar time bounds. There is no study that explicitly considers the computation of the k-edge-connected or the k-vertex-connected subgraphs of undirected graphs, however, the problem can be reduced to the problem on directed graphs in a straightforward manner. Furthermore, for undirected graphs the runtime (which is implied by [8], see below) of the basic algorithm for k-edge-connected subgraphs for constant k can be reduced to $O(n^2 \log n)$ by additionally maintaining a sparse certificate [5, 26, 30].

k-connected components in digraphs. Very recently Georgiadis et al. [14, 13] showed that the 2-edge-connected and the 2-vertex-connected components of a directed graph can be computed in linear time. Nagamochi and Watanabe [27] gave an $O(kmn)$ time algorithm for computing the k-edge-connected components in directed graphs.

k-edge-connected subgraphs in digraphs. A simple algorithm for computing the maximal 2-edge-connected subgraphs is to remove at least one strong bridge of a strongly connected component of the graph and repeat on the resulting graph. It is known since 1976 how to compute a strong bridge [29] in $O(m + n \log n)$ time, and since 1985 in $O(m)$ time [10], resulting in an $O(mn)$ time algorithm for computing the 2-edge-connected subgraphs of a directed graph. Recently, Italiano et al. [20] gave a linear time algorithm for computing *all* strong bridges of a directed graph in $O(m)$ time, of which there can be $O(n)$ many. A similar idea can be used to compute the k-edge-connected subgraphs. In this case, in each iteration we remove the minimum edge cut of each strongly connected component of the graph, if its size does not exceed $k - 1$. Since an edge cut of size k can be computed in time $O(km \log n)$ [8], and in each iteration we disconnect at least one pair of vertices, this algorithm runs in $O(kmn \log n)$ time. Recently, Henzinger et al. [17] presented an $O(n^2)$ time algorithm for computing the 2-edge-connected subgraphs of a directed graph and an $O(n^2 \log n)$ time algorithm for the k-edge-connected subgraphs for any constant k. Their algorithm uses a sparsification technique introduced in [1, 18] that can be used, under appropriate structural properties, to replace a factor of m in the running time of an algorithm by n.

k-vertex-connected subgraphs in digraphs. 2-vertex-connected subgraphs were first studied in 1980 by Erusalimskii and Svetlov [6], but they did not analyze the running time of their algorithm. Very recently, Jaberi [21] showed that their algorithm runs in $O(nm^2)$ time and presented an $O(mn)$ time algorithm. Prior to Jaberi, Makino [25] gave an algorithm for computing the maximal k-vertex-connected subgraphs of a directed graph in time $O(n \cdot \mathcal{S})$, where $\mathcal{S}$ is the running time for computing a single vertex cut of size at most $k - 1$. Since one strong articulation point [12], or even all the strong articulation points [20], can be computed in linear time, Makino's algorithm can be implemented so as to compute the 2-vertex-connected subgraphs of a directed graph in time $O(mn)$. Combined with Gabow's algorithm for identifying k-vertex cuts [9],

Makino's algorithm yields a runtime of $O(mn \cdot (n + \min\{k^{5/2}, kn^{3/4}\}))$ for k-vertex-connected subgraphs; an $O(kmn^2)$ time algorithm is already implied by combining it with [7]. The recent algorithm of Henzinger et al. [17] computes the 2-vertex-connected subgraphs in time $O(n^2)$ and extends to the k-vertex-connected subgraphs for constant k with a runtime of $O(n^3)$.

Key Ideas. We next outline the main ideas behind our approach. The basic algorithm for 2-edge-connected subgraphs can be seen as maintaining a partition of the vertices that is iteratively refined by identifying parts that cannot be in the same 2-edge-connected subgraph, which are then separated from each other in the maintained partition. In the basic algorithm these parts are identified by computing bridges and SCCs. The main technical contribution of this work is a subroutine that can identify a "small" part that can be separated from the rest of the graph by local depth-first searches that, starting from one given vertex, explore only the edges in this small part and a proportional number of edges outside of it.

For 2-edge-connected subgraphs we call the subgraphs identified in this way 1-*edge-out* and 1-*edge-in* components[1]. A k-edge-out (resp., k-edge-in) component of a vertex u is a subgraph that contains u and has at most k edges from (resp., to) the subgraph to (resp., from) the rest of the graph. We start the searches for these subgraphs from all vertices that have lost edges since the last time bridges and SCCs were computed and only recompute bridges and SCCs when no 1-edge-out or 1-edge-in component with at most $\sqrt{m}$ edges exists[2].

The intuition for the local depth-first searches for edge connectivity can be better understood in terms of maximum flow in uncapacitated graphs. Assume there is a 1-edge-out component of a vertex u. Since this subgraph has at most one outgoing edge to the rest of the graph, the vertex u can send at most one unit of flow to any vertex outside of the subgraph. Thus if we find a path along which we can send one unit of flow to some vertex outside of the subgraph and then look at the residual graph given this flow, then there is no edge from the subgraph to the rest of the graph in the residual graph. We find such a flow using depth-first search and then use a second search to explore the subgraph that is still reachable from u in the residual graph.

Finding $k-1$ paths to send flow out of a $(k-1)$-edge-out component is more difficult for $k > 2$. We show that one can exploit the properties of depth-first search to find a set of $O(k)$ paths of which at least one of them leaves the $(k-1)$-edge-out component. As we have to do this for k many searches, each conducted in the residual graph after the previous search, this yields an exponential dependence on k. For any constant $k > 2$ we compute the k-edge-connected subgraphs in time $O(m^{3/2} \log n)$ time, where the additional factor of $\log n$ compared to $k = 2$ is due to the increased cost of computing cuts with at most $k-1$ edges.

The notion of a k-edge-out (resp., k-edge-in) component of a vertex u is adjusted to vertex connectivity as follows. A k-vertex-out (resp., k-vertex-in) component S of a

[1]A similar notion called 2-isolated set was introduced in [17].

[2]A similar overall algorithmic structure was used in [17, Appendix B] and, for a different problem, e.g., in [1].

vertex u is a subgraph that contains u and at most k vertices in the subgraph have edges from (resp., to) the subgraph to (resp., from) the rest of the graph.

For vertex connectivity some additional difficulties arise. First, the 2-vertex-connected subgraphs partition the edges rather than the vertices (apart from degenerate cases), i.e., when we find a strong articulation point and run our algorithm recursively on the subgraphs that it separates, the strong articulation point is included in each of these subgraphs. Second, the intuition of flows and residual graphs cannot be applied directly; instead, we let one depth-first search "block" specific vertices (those whose DFS subtree is adjacent to many edges) and let a second search "unblock" vertices such that it can explore the 1-vertex-out component but not the remaining graph.

1.1 Preliminaries

For a directed graph G we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. The reverse graph of a directed graph $G = (V, E)$, denoted by $G^R = (V, E^R)$, is the directed graph that results from G after reversing the direction of all edges. By $G \setminus S$ and $G \setminus Q$ we denote the graph G after the deletion of a set S of vertices and after the deletion of a set Q of edges, respectively. We refer to the subgraph of G induced by the set of vertices S as $G[S]$. Let H be a strongly connected graph, or a strongly connected component of some larger graph. We say that deleting a set of edges Q (resp., set of vertices S) *disconnects* H, if $H \setminus Q$ (resp., $H \setminus S$) is not strongly connected. Given a set of vertices C, we say that a set of edges Q (resp., a set of vertices S) disconnects C from the rest of the graph if there is no pair of vertices $(x, y) \in C \times (V \setminus C)$ that are strongly connected in $G \setminus Q$ (resp., $G \setminus S$). For the sake of simplicity, we write $S \subseteq G$, instead of $S \subseteq V(G)$, to denote that a set of vertices S is a subset of the vertices of a graph G. We similarly write $Q \subseteq G$ instead of $Q \subseteq E(G)$, where Q is a subset of the edges of the graph G. Furthermore, we write $v \in G$ and $e \in G$ instead of $v \in V(G)$ and $e \in E(G)$, respectively.

We use the term tree to refer to a rooted tree with edges directed away from the root. Given a tree T, a vertex u is an ancestor (resp., descendant) of a vertex v if there is a directed path from u to v (resp., from v to u) in T. We denote by $T[u, v]$ the path from u to v in T. We use $T(u)$ to denote the set of vertices that are descendants of u in T.

There is a natural connection between edge cuts and maximum flow in unweighted graphs. The maximum flow that can be sent from a source vertex s to a target vertex t in directed graphs with uncapacitated edges is equal to the number of edge-disjoint paths directed from s to t. Therefore, the existence of a cut consisting of k edges directed from a set of vertices A to a set of vertices B implies that the maximum flow that can be pushed from any vertex in A to any vertex in B is at most k. We implicitly use this connection between edge cuts and max flow. We further assume that the reader is familiar with depth-first search (DFS), see, e.g., [28].

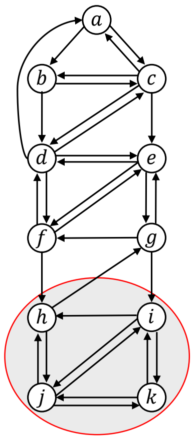


Figure 2: An example of a 1-edge-out component of j.

2 Maximal 2-edge-connected subgraphs of a digraph

In this section we first show how to identify 1-edge-out components that contain at most Δ edges in time proportional to Δ. Applied to the reverse graph, the same algorithm finds 1-edge-in components. We then use this subroutine with $\Delta = \sqrt{m}$ to obtain an $O(m^{3/2})$ algorithm for computing the maximal 2-edge-connected subgraphs of a given directed graph.

2.1 1-edge-out and 1-edge-in components

Definition 1. *Let $G = (V, E)$ be a digraph and $u \in V$ be a vertex. A k-edge-out component of u is a minimal subgraph S of G that contains u and has at most k outgoing edges to $G \setminus S$.*

We similarly define a k-edge-in component of u.

Definition 2. *Let $G = (V, E)$ be a digraph and $u \in V$ be a vertex. A k-edge-in component of u is a minimal subgraph S of G that contains u and has at most k incoming edges from $G \setminus S$.*

See Figure 2 for an example of a k-edge-cut and Figure 3 for an example of a k-edge-in with $k = 1$. Note that u may have more than one k-edge-out (resp., k-edge-in) component. Also note that for $k' < k$, every k'-edge-out component of u is a k-edge-out component of u as well. For the case when $k = 1$, the outgoing (resp., incoming) edge of a 1-edge-out (resp., 1-edge-in) component S is either a strong bridge or an edge between strongly connected components of the graph. Moreover, each 2-edge-connected subgraph is either completely contained in S or in $G \setminus S$ (see also [17]).

We next present an algorithm that takes as input a graph G, a vertex $u \in V(G)$, and a parameter $\Delta < m/2$, and that spends time at most $O(\Delta)$ to search for a 1-edge-out

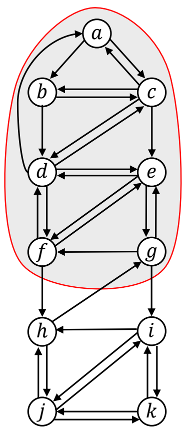


Figure 3: An example of a 1-edge-in component of f.

component of u in G. The algorithm may fail to find such a component, and we therefore prove the following guarantees about its outcome:

- If u has a 1-edge-out component with at most Δ edges, then the algorithm returns a 1-edge-out component for u with at most 2Δ edges.
- If every 1-edge-out component of u has more than Δ edges, then the algorithm may return a 1-edge-out component for u with at most 2Δ edges, but it may also return the empty set (i.e., fail to find a 1-edge-out component for u).

Note that by using exponential search in Δ, the algorithm can find a 1-edge-out component for a given vertex u in time that is linear in the number of edges of the smallest 1-edge-out component that contains u. For our purpose in Section 2.2, however, it suffices to distinguish between small and large 1-edge-out components and only use one fixed choice of Δ. We thus use the algorithm to quickly find a small 1-edge-out component S, given a vertex u in S.

For the rest of this section, we assume that the starting vertex u can reach at least $2\Delta + 1$ edges. Notice that if u cannot reach $2\Delta + 1$ edges, then the reachable subgraph from u defines a 0-edge-out component of u containing at most 2Δ edges. In this case, the algorithm returns this 0-edge-out component. We use exactly the same algorithm executed on the reverse graph for 1-edge-in components and therefore only describe the algorithm for 1-edge-out components. First, we provide the following supporting lemmas.

Lemma 3. *Let (x, y) be the outgoing edge of a 1-edge-out component $1EOut(u)$ of a vertex u. Then u has a path to every vertex $v \in 1EOut(u)$ that is contained entirely within the subgraph $1EOut(u)$. Moreover, u has two edge-disjoint paths to x within $1EOut(u)$.*

Proof. We begin by showing that u has a path to every vertex $v \in 1EOut(u)$ that is contained entirely within the subgraph $1EOut(u)$. Assume for the sake of contradiction that there is a set of vertices $C \subset 1EOut(u)$ such that the vertices of C are unreachable from u in $1EOut(u)$. Then there is no edge (w, z) with $w \in 1EOut(u) \setminus C$ and $z \in C$ and thus the only possible outgoing edge from $1EOut(u) \setminus C$ is (x, y). Thus, $1EOut(u) \setminus C$ is a 1-edge-out component of u, which contradicts the minimality of $1EOut(u)$.

We now show that u has two edge-disjoint paths to x in $1EOut(u)$. First, we note that all simple paths from u to x contain only vertices in $1EOut(u)$ since there is no edge $(x', y') \neq (x, y)$ leaving $1EOut(u)$. Assume, for the sake of contradiction, that all paths from u to x in $1EOut(u)$ share a common edge (w, z). Then, u does not have a path to z in $1EOut(u) \setminus (w, z)$. Let $C \subset 1EOut(u)$ be the set of vertices that become unreachable from u in $1EOut(u) \setminus (w, z)$. (Notice that $|C| \geq 1$ since $z \in C$.) Clearly, there is no edge (w', z') such that $w' \in V(1EOut(u)) \setminus C$ and $z' \in C$. Hence, the only outgoing edge from $1EOut(u) \setminus C$ is (w, z). Thus, $1EOut(u) \setminus C$ is a 1-edge-out component of u, which again contradicts the minimality of $1EOut(u)$. □

Our algorithm starts a DFS traversal F_1 from u. We charge to a visited vertex its outgoing edges that were discovered by F_1. We stop F_1 when the number of traversed edges reaches $2\Delta + 1$. Let T be the DFS tree constructed by the DFS traversal. We define the weight of a vertex v, denoted by $w(v)$, to be the total number of edges charged to the descendants of v in T (including v). Assume u is contained in a 1-edge-out component C with at most Δ edges. Note that for any vertex $v \neq u$ whose DFS subtree only explores edges inside C (that is, the DFS search never leaves C after vising v), we have $w(v) < \Delta$.

Lemma 4. *Let $1EOut(u)$ be a 1-edge-out component of u such that $|E(1EOut(u))| \leq \Delta$, let (x, y) be the outgoing edge of $1EOut(u)$, and let T be a DFS tree of a DFS traversal from u that visited $2\Delta + 1$ edges. Then $w(v) \geq \Delta$ for each vertex v on the path from u to y in T, i.e., $v \in T[u, y]$, and $w(v) < \Delta$ for each $v \in 1EOut(u) \setminus T[u, x]$.*

Proof. Since $|E(1EOut(u))| \leq \Delta$ and the DFS traversal visits at least $2\Delta + 1$ edges, it follows that T visits at least Δ edges in the subtraversal from y (that excludes (x, y)). Therefore, for each $v \in T[u, y]$ it holds that $w(v) \geq \Delta$. Note that (x, y) can be used only once by the DFS traversal, and any traversal from u that does not visit vertices outside $1EOut(u)$ cannot reach more than Δ edges. Further, since the DFS starts from u, any vertex apart from u can only reach strictly less than Δ edges without its subtraversal leaving $1EOut(u)$. None of the subtraversals from vertices $v \in 1EOut(u) \setminus T[u, y]$ could visit vertices outside $1EOut(u)$, since either they were visited after the edge (x, y), or they could not visit y. Thus, for each vertex $v \in 1EOut(u) \setminus T[u, x]$, it holds that $w(v) < \Delta$. □

Lemma 5. *Let F be a DFS traversal that visited $2\Delta + 1$ edges and let T be the DFS tree generated by F. The edges $e = (x, y) \in T$ with $w(y) \geq \Delta$ form a path in T.*

Proof. Assume by contradiction that there are two tree edges $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ with $w(y_1) \geq \Delta$ and $w(y_2) \geq \Delta$ that do not have an ancestor-descendant relation in T (i.e., y_1 is not an ancestor of x_2 and y_2 is not an ancestor of x_1). Since also the edges (x_1, y_1) and (x_2, y_2) are visited by T, this contradicts the fact that the traversal visited $2\Delta + 1$ edges. Therefore, all edges $e = (x, y) \in T$ with $w(y) \geq \Delta$ form a path in T. □

After the execution of the first DFS F_1, by Lemma 5, there is a path P of T such that we have $w(y) \geq \Delta$ for every edge $e = (x, y)$ of P. We call this path the *heavy path* of F_1, and the edges contained in the heavy path the *heavy edges* of F_1. Note that (1) the heavy path has to leave a 1-edge-out component of u with at most Δ edges for the search to reach more than Δ edges and (2) the heavy path cannot enter the component again after leaving it because the subtree of any incoming edge of the component cannot contain Δ or more edges as the only outgoing edge of the component was already used. We construct the residual graph G' formed from G by reversing the direction of the heavy edges of F_1. The residual graph will be used as follows. If there exists a 1-edge-out component $1EOut(u)$ of u containing at most Δ edges, then the heavy path P can be interpreted as sending one unit of flow out of $1EOut(u)$ and in the residual graph with respect to this flow no additional unit of flow can be sent out of $1EOut(u)$. That means that no other search from u is able to have an outgoing path from $1EOut(u)$. Next, we execute a second traversal F_2 from u (not necessarily a depth-first search) on G'. We show that if there exists a 1-edge-out component $1EOut(u)$ of u containing at most Δ edges, this second traversal has two main properties: (i) it never visits edges outside of $G'[V(1EOut(u))]$, and (ii) it visits all the edges in $G'[V(1EOut(u))]$. Whenever F_2 traverses more than Δ edges, we terminate the search and conclude that any 1-edge-out component of u contains more than Δ edges.

Lemma 6. *Let G' be the residual graph obtained from G by reversing the direction of the heavy edges of F_1. The traversal F_2 reaches at most Δ edges in G' if and only if there exists a 1-edge-out component $1EOut(u)$ of u containing at most Δ edges. Moreover, if F_2 traverses at most Δ edges, then the subgraph in G induced by the vertices traversed by F_2 defines $1EOut(u)$.*

Proof. Let us first assume that there exists a 1-edge-out component $1EOut(u)$ of u that contains at most Δ edges and has one outgoing edge (x, y). By Lemma 4, the edge (x, y) is reversed in the residual graph G'. Moreover, the lemma implies that no incoming edge to $1EOut(u)$ is reversed in G' because each incoming edge (v, z) either has $w(z) < \Delta$ or $z \in T[u, x]$; in the latter case (v, z) cannot be a DFS tree edge. Thus, $G'[V(1EOut(u))]$ has no outgoing edges to $G'[V(G) \setminus V(1EOut(u))]$. Therefore, F_2 cannot visit more than Δ edges. We now show that F_2 visits all vertices in $G'[V(1EOut(u))]$ using only paths internal to $G'[V(1EOut(u))]$. Notice that this does not trivially follow from Lemma 3 since we are operating on the residual graph G', where the direction of some edges of $1EOut(u)$ is reversed. Assume by contradiction that u cannot visit all vertices in $G'[V(1EOut(u))]$. Then, there is a set of vertices $C \subset V(1EOut(u))$ that has no incoming edge in the residual graph G'. By Lemma 5 the edges that are reversed in the residual graph G' form a path P in the DFS tree of F_1. The path P contains an incoming edge to C in G since otherwise $1EOut(u) \setminus C$ is a 1-edge-out component of u, contradicting the minimality of $1EOut(u)$. Since C has no incoming edges from $1EOut(u) \setminus C$ in G', we have that P has no outgoing edges from C to $1EOut(u) \setminus C$. Therefore $T[u, y] \in P$ implies $x \in C$. Since P does not enter $1EOut(u)$ after leaving through (x, y), only one edge incident to C was reversed in G'. As there is no edge incident to C in G', this is a contradiction to Lemma 3, which says that u has two edge-disjoint paths to x. Hence no such set C exists and F_2 traverses all vertices of $1EOut(u)$.

Now we show the opposite direction. Assume that F_2 visits at most Δ edges in the residual graphs. We will show that there exists a 1-edge-out component $1EOut(u)$ of u that contains at most Δ edges and that is given by the subgraph induced by the vertices traversed by F_2. Let C be the subgraph that F_2 traversed in the residual graph. Then C has no outgoing edges in G', since otherwise their neighbors would also be traversed by F_2. Since F_1 visited $2\Delta + 1$ edges, there is at least one edge e^* incoming to C in G' that was reversed. Note that there cannot exist more than one incoming edge to C in G' that was reversed after F_1, since that would imply the existence of an outgoing edge from C since the set of reversed edges forms a path by Lemma 5. Hence u has no path to any of the vertices in $V \setminus C$ in the residual graph G', and has only one outgoing edge in the original graph G. Therefore, after restoring the reversed edges, C forms a 1-edge-out component of u that contains at most Δ edges, with the only outgoing edge being e^*. Notice that the vertices of C were all traversed by F_2. It remains to show that there is no 1-edge-out component $1EOut'(u)$ of u with one outgoing edge (x', y') and such that $1EOut'(u) \subset 1EOut(u)$. Assume by contradiction that there exists such a component. By Lemma 4 the traversal F_1 reversed (x', y'), and there is no other outgoing edge from $1EOut'(u)$ in the residual graph. Therefore, F_2 cannot visit vertices outside $1EOut'(u)$. A contradiction to the fact that F_2 visited all the edges and vertices in $1EOut(u)$. □

Recall that we assumed in the beginning of this section that u reaches at least $2\Delta + 1$ edges. That allows us to eliminate the existence of a 0-edge-out component of u with at most 2Δ edges. This property is important for determining whether there exists a 1-edge-out component of u with at most Δ edges, since our algorithm uses a DFS search (namely F_1) to visit $2\Delta + 1$ edges. After the execution of the traversal F_2 on the residual graph G', we can answer whether there exists a 1-edge-out component of u with at most Δ edges, as shown in Lemma 6. The pseudocode of our algorithm is illustrated in Procedure 1EdgeOut. The following lemma summarizes the result of this section.

Lemma 7. *Procedure 1EdgeOut computes a* 1*-edge-out* (*resp.,* 1*-edge-in*) *component of* u *with at most* 2Δ *edges or decides that there is no* 1*-edge-out* (*resp.,* 1*-edge-in*) *component of* u *with at most* Δ *edges. Moreover, Procedure 1EdgeOut runs in* $O(\Delta)$ *time.*

2.2 Computing the 2-edge-connected subgraphs.

Let $G = (V, E)$ be a digraph. A straightforward algorithm for computing the 2-edge-connected subgraphs is to recursively remove, from G, one strong bridge of each strongly connected component of G until no strong bridges can be found. In each recursive call at least one vertex gets disconnected from the rest of the graph. Since computing the strongly connected components and one strong bridge (or all strong bridges) of a digraph can be done in linear time, this simple algorithm runs in $O(mn)$ time.

```
Procedure 1EdgeOut($G$, $u$, $\Delta$)
  Input: Digraph $G = (V, E)$, a vertex $u$, and an integer $\Delta$
  Output: Either a 1-edge-out component of $u$ with at most $2\Delta$ edges or $\emptyset$; if $\emptyset$ is
          returned, then every 1-edge-out component that contains $u$ has more
          than $\Delta$ edges
1  Execute DFS $F_1$ from $u$ for up to $2\Delta + 1$ edges
2  Let $S_1$ be the vertices reached by $F_1$
3  if $F_1$ cannot reach $2\Delta + 1$ edges then
4      return $G[S_1]$ as 1-edge-out component of $u$
5  else
6      Let $P$ be the heavy path of $F_1$
7      Let $G'$ be $G$ after reversing the direction of the edges of $P$
8      Execute DFS $F_2$ from $u$ on $G'$ for up to $\Delta + 1$ edges
9      Let $S_2$ be the vertices reached by $F_2$
10     if $F_2$ cannot reach $\Delta + 1$ edges then
11         return $G[S_2]$ as 1-edge-out comp. of $u$
12     else
13         return $\emptyset$
```

In our algorithm we build on the simple algorithm described above. The high-level idea of our approach is to (a) find subgraphs with at most $\sqrt{m}$ edges that are not 2-edge-connected to the rest of the graph in total time $O(m\sqrt{m})$ and by this (b) limit the maximum recursion depth to $\sqrt{m}$ by only making recursive calls when large subgraphs will be disconnected from each other or the remaining graph has at most $O(\sqrt{m})$ edges. This is done as follows. We use the terms small and large components to refer to subgraphs that contain at most and more than $\sqrt{m}$ edges, respectively. We first identify all the small components that can be disconnected from the rest of the graph by a single edge deletion. In each recursive call of the algorithm we maintain a list L of vertices for which we want to identify small 1-edge-out and 1-edge-in components. Initially, we set the list L to contain all vertices in order to find all small components that can be separated by at most one edge. We search for such small subgraphs using the algorithm from Section 2.1. We compute 1-edge-in components by executing $1\mathit{EdgeOut}(G^R, u, \sqrt{m})$, where G^R is the reverse graph of G. Whenever we find a small 1-edge-out or 1-edge-in component, we remove all its incident edges and search for more small 1-edge-out or 1-edge-in components in the remaining graph. We do that by inserting the endpoints of the deleted edges into the list L. If, on the other hand, we cannot find new small components, we conclude that either the remaining graph is 2-edge-connected or there are at least two large sets of vertices that will get disconnected by either recomputing SCCs or by the removal of a strong bridge. In a final phase of each recursive call we compute the SCCs of the graph and for each SCC we remove one strong bridge and then recursively call the algorithm on every resulting SCC. Before each recursive call, we

Algorithm 1: $2ECS(G, L)$

```
Input: A strongly connected digraph $G = (V, E)$ and a list of vertices $L$ (initially $L = V$)
Output: The 2-edge-connected subgraphs of $G$
1  Let $m_0$ be number of edges of initial graph
2  if $G$ has no strong bridge then
3      return $\{G\}$ as 2-edge-connected subgraph
4  while $L \neq \emptyset$ & $G$ has more than $2\sqrt{m_0}$ edges do
5      Extract a vertex $u$ from $L$
6      $S \leftarrow$1EdgeOut($G$, $u$, $\sqrt{m_0}$)
7      $S^R \leftarrow$1EdgeOut($G^R$, $u$, $\sqrt{m_0}$)
8      If either $S$ or $S^R$ is not empty, remove from $G$ all edges incident to one
       non-empty set of $S$ and $S^R$ and add their endpoints to $L$
9  Compute SCCs $C_1, \ldots, C_c$ of $G$
10 $U \leftarrow \emptyset$
11 foreach $C_i, 1 \leq i \leq c$ do
12     Remove one strong bridge from $C_i$
13     Recompute SCCs and delete the edges between them
14     foreach SCC $C'$ do
15         Insert into $L'$ the vertices of $C'$ that are endpoints of newly deleted edges
16         $U \leftarrow U \cup 2ECS(C', L')$
17 return $U$
```

initialize the lists L to contain the vertices that lost an edge during the last phase of the parent recursive call. We keep this list in order to restrict the total number of searches for small separable components to $O(m + n)$ since, after initially adding all vertices to the list of the initial call, we only add the endpoints of deleted edges into the lists (which is $O(m)$). Algorithm 1 contains the pseudocode of our algorithm.

The following is a key property that allows us to find small sets that are not strongly connected or that can be disconnected by deleting a single edge, or to conclude that there are no such small sets. Every new 1-edge-out component that appears in the graph throughout the algorithm must have lost an outgoing edge. Respectively, every new 1-edge-in component that appears must have lost an incoming edge. Therefore, we use the list L to keep track of the vertices that have lost an edge and for each such vertex u we search for new small 1-edge-out or 1-edge-in components of u. If no such small components exist in a set of vertices C, then we know that either C is a 2-edge-connected subgraph or either recomputing SCCs or the deletion of some strong bridge disconnects at least two large components. These properties are summarized in the following lemma.

Lemma 8. *Let C be a set of vertices in G. Every 1-edge-out or 1-edge-in component (of some vertex $u \in C$) in $G[C]$ that is not such a component in G must contain an endpoint* *of an edge incident to $G[C]$. Moreover, if there is no 1-edge-out or 1-edge-in component containing at most $\sqrt{m}$ edges for any vertex $u \in C$ in both G and $G[C]$, then one of the following holds:*

(*a*) *$G[C]$ is a 2-edge-connected subgraph of G.*

(*b*) *There are two sets $A, B \subset C$ with $|E(G[A])|, |E(G[B])| > \sqrt{m}$ such that A and B are in different strongly connected components of $G[C]$.*

(*c*) *For each strong bridge of $G[C]$ there are two sets $A, B \subset C$ with $|E(G[A])|, |E(G[B])| > \sqrt{m}$ that get disconnected by the deletion of the strong bridge.*

Proof. We first show that every 1-edge-out component $1EOut(u)$ of some vertex $u \in C$ that is no 1-edge-out component in G must contain a vertex $x \in 1EOut(u)$ such that there is an edge (x, y) with $y \notin C$. Assume, by contradiction, that $1EOut(u)$ exists but there is no such edge (x, y) in G with $x \in 1EOut(u)$ and $y \notin C$. In this case we have that the very same component $1EOut(u)$ is a 1-edge-out component of u in G. The same argument on the reverse graph shows that every new 1-edge-in component (of some vertex $u \in C$) in $G[C]$ must contain an endpoint of an edge incident to $G[C]$ in G.

We now turn to the second part of the lemma. If $G[C]$ is strongly connected and does not contain a strong bridge, then $G[C]$ is 2-edge-connected and thus (a) holds. If $G[C]$ is not strongly connected, then it contains (at least) two disjoint sets $A, B \subset C$ such that both $G[A]$ and $G[B]$ are strongly connected components of $G[C]$ and $G[A]$ has no outgoing edge in $G[C]$ (i.e., $G[A]$ is a sink in the DAG of SCCs of $G[C]$) and $G[B]$ has no incoming edge in $G[C]$ (i.e., $G[B]$ is a source in the DAG of SCCs of $G[C]$). That is, in $G[C]$ we have that $G[A]$ is a 1-edge-out component of some $u \in C$ and $G[B]$ is a 1-edge-in component of some $u' \in C$. Both can have the same property in G or be new such components in $G[C]$ compared to G. In any case it contradicts the assumptions if one of them has at most $\sqrt{m}$ edges and otherwise statement (b) holds. If $G[C]$ is strongly connected and contains a strong bridge e^*, an analogous argument can be made for two disjoint sets $A, B \subset C$ by considering the DAG of SCCs of $G[C] \setminus e^*$. In this case e^* is the only incoming edge of B and the only outgoing edge of A in $G[C]$. We have that case (c) holds if the assumptions of the lemma are satisfied. □

Lemma 9. *Algorithm 2ECS runs in $O(m\sqrt{m})$ time.*

Proof. First notice that each time we search for a 1-edge-out or a 1-edge-in component, we are searching for a component with one outgoing (resp., incoming) edge containing at most $\sqrt{m}$ edges or with no outgoing (resp., incoming) edges and at most $2\sqrt{m}$ edges. We can identify if such a component containing a given vertex u exists in time $O(\sqrt{m})$ by using the algorithm of Section 2.1. We initiate such a search from each vertex that appears in the list L of some recursive call of the algorithm. Initially, we place all vertices in the list L. Throughout the algorithm we insert into L only vertices that are endpoints of deleted edges. Therefore, the number of vertices that are added to the lists L throughout the algorithm is $O(m)$. Hence, the total time spent on these searches is $O(m\sqrt{m})$.

Consider now the time spend in each recursive call without the searches for 1-edge-out and 1-edge-in components. Let G' be the graph for which the recursive call is made and let $m_{G'} = |E(G')|$. In each recursive call the algorithm spends $O(m_{G'})$ time searching for strong bridges in G' in lines 2 and 12 and computing SCCs in lines 9 and 13. Since the subgraphs of different recursive calls at the same recursion depth are disjoint, the total time spent at each level of the recursion is $O(m)$. We now bound the recursion depth with $O(\sqrt{m})$.

We show that the graph passed to each recursive call has at most $\max\{m_{G'} - \sqrt{m}, 2\sqrt{m}\}$ edges, or G' is a 2-edge-connected subgraph and thus the recursion stops. This implies a recursion depth of $O(\sqrt{m})$ as follows. If the graph passed to a recursive call has at most $2\sqrt{m}$ edges, then also the number of vertices of this graph is at most $2\sqrt{m}$. Therefore, even if the algorithm only removes one strong bridge from every strongly connected component in each recursive call, the total recursion depth is at most $O(\sqrt{m})$. On the other hand, the number of times that the graph passed to a recursive call has $\sqrt{m}$ fewer edges than G' is at most $\sqrt{m}$. Overall, this implies that the recursion depth is bounded by $O(\sqrt{m})$.

It remains to show the claimed bound on the size of the graph passed to a recursive call in line 16. For every 1-edge-out or 1-edge-in component with at most $2\sqrt{m}$ edges that is discovered throughout the algorithm, its incident edges are removed and therefore it will be in a separate strongly connected component with at most $2\sqrt{m}$ edges. Let C be the set of vertices that were not included in any 1-edge-out or 1-edge-in component. By Lemma 8 the subgraph $G'[C]$ either is a 2-edge-connected subgraph or there are two sets A and B with $|E(A)|, |E(B)| > \sqrt{m}$ that will be separated in Line 12. Thus, every graph passed to the recursive call will have at most $\max\{|E(G')| - \sqrt{m}, 2\sqrt{m}\}$ edges. The lemma follows. $\square$

Lemma 10. *Algorithm 2ECS is correct.*

Proof. First note that by assumption the initial call to the algorithm is on a strongly connected graph and that recursive calls are only made on strongly connected subgraphs. Thus whenever the algorithm reports a 2-edge-connected subgraph in line 3, then it is a strongly connected subgraph that does not contain any strong bridges, which is by definition a 2-edge-connected subgraph. Thus it suffices to show that the algorithm reports all the maximal 2-edge-connected subgraphs. Notice that this also implies that the reported 2-edge-connected subgraphs are maximal. Let C be a maximal 2-edge-connected subgraph. We show that the vertices of C do not get separated by the algorithm, and therefore C is reported eventually as a 2-edge-connected subgraph. Since there are two edge-disjoint paths between every pair of vertices in C, any search for either a 1-edge-out or a 1-edge-in component of a vertex u (lines 6–7) either returns a superset of C or fails to identify such a set containing a subset of the vertices of C. Furthermore, notice that any deletion of an edge that does not have both endpoints in C does not affect the fact that C is 2-edge-connected. That is, unless an edge with both endpoints in C is deleted, no strong bridge appears in C. Thus, it remains to show that no edge (x, y) such that $x, y \in C$ is ever deleted throughout the algorithm. The edges deleted in line 8 of the algorithm are incident to a 1-edge-out or a 1-edge-in component. Since C is always fully inside or fully outside of such a set, no edge from C is deleted. The edges deleted in line 12 are strong bridges and the edges deleted in line 13 before the recursive calls are between separate strongly connected components. Since C is 2-edge-connected, no edges from C are deleted. Finally, notice that at each level of recursion at least one of the strong bridges of each strongly connected component of the graph is deleted and the algorithm is recursively executed in each resulting strongly connected component. Thus, finally there will be a recursive call for each strongly connected subgraph that does not contain strong bridges, including C. $\square$

We have shown the following theorem.

Theorem 11. *The maximal* 2*-edge-connected subgraphs of a digraph can be computed in* $O(m^{3/2})$ *time.*

3 Maximal 2-vertex-connected subgraphs in directed graphs

In this section we first introduce a procedure for identifying 1-vertex-out components containing at most Δ edges in time proportional to Δ. The same algorithm applied to the reverse graph identifies 1-vertex-in components. We then use this subroutine with $\Delta = \sqrt{m}$ to obtain a $O(m^{3/2})$ algorithm for computing the maximal 2-vertex-connected subgraphs of a given directed graph.

3.1 1-vertex-out and 1-vertex-in components.

We begin with the definition of 1-vertex-out and 1-vertex-in components of a vertex.

Definition 12. *Let* $G = (V, E)$ *be a digraph and* $u \in V$ *be a vertex. A* k*-vertex-out component of* u *is a minimal subgraph* S *of* G *that contains* u *and has at most* k *vertices* $X \subset V(S)$, $u \notin X$, *with outgoing edges to* $G \setminus S$.

Definition 13. *Let* $G = (V, E)$ *be a digraph and* $u \in V$ *be a vertex. A* k*-vertex-in component of* u *is a minimal subgraph* S *of* G *that contains* u *and has at most* k *vertices* $X \subset V(S)$, $u \notin X$, *with incoming edges from* $G \setminus S$.

As in the case of k-edge-out (resp., k-edge-in) components, a vertex u may have more than one k-vertex-out (resp., k-vertex-in) component. Also note that for $k' < k$, every k'-vertex-out component of u is a k-vertex-out component of u as well. For the case when $k = 1$, the only vertex x that has outgoing (resp., incoming) edges from a 1-vertex-out (resp., 1-vertex-in) component S is either a strong articulation point or a vertex that has outgoing (resp., incoming) edges to vertices that belong to different strongly connected components than x. Moreover, each 2-vertex-connected subgraph is either completely contained in S or in $(G \setminus S) \cup x$.

For a given vertex u and a parameter $\Delta < m/2$, we present an algorithm for computing a 1-vertex-out component of u that runs in time $O(\Delta)$ and has the following guarantees:

- If there exists a 1-vertex-out component of u with at most Δ edges, then it returns a 1-vertex-out component of u with at most 2Δ edges.
- If no 1-vertex-out component with at most Δ edges exists, it might either return a 1-vertex-out component of u with at most 2Δ edges or the empty set.

As mentioned earlier, our algorithm identifies a 1-vertex-out component of u in time proportional to its size (i.e., its number of edges). In Section 3.2 we will use this algorithm to determine quickly whether there exist 1-vertex-out (resp., 1-vertex-in) components of small size (namely, containing at most a predefined number of edges Δ), or conclude that all 1-vertex-out (resp., 1-vertex-in) components have large size. We show that this is sufficient to bound the total running time of our algorithm for computing the 2-vertex-connected subgraphs.

For the rest of this section, we assume that we are given a starting vertex u that can reach at least $2\Delta + 1$ edges. If this is not the case, then the reachable subgraph from u defines a valid 1-vertex-out component of u that contains at most 2Δ edges and has no outgoing edges. The exactly same algorithm executed on the reverse graph computes a 1-vertex-in component of u that contains at most 2Δ edges, or we conclude that there is no 1-vertex-in component of u with at most Δ edges. Since the algorithm for computing a 1-vertex-in component of u is identical to the algorithm for computing a 1-vertex-out component of u when executed on the reverse graph, we only describe the algorithm for finding 1-vertex-out components. The following supporting lemma shows important properties which we exploit in our algorithm.

Lemma 14. *Let $1VOut(u)$ be a 1-vertex-out component of a vertex u and let x, $x \neq u$, be the only vertex having outgoing edges from $1VOut(u)$. It holds that u has a path to every vertex $v \in 1VOut(u)$ that is contained entirely within the subgraph $1VOut(u)$. Moreover, u has two internally vertex-disjoint paths to x in $1VOut(u)$.*

Proof. We begin by showing that u has a path to every vertex $v \in 1VOut(u)$ that is contained entirely within the subgraph $1VOut(u)$. Assume, for the sake of contradiction, that there is a set of vertices C such that the vertices of C are unreachable from u in $1VOut(u)$. Then there is no edge (w, z) where $w \in 1VOut(u) \setminus C$ and $z \in C$ and thus the outgoing edges from the vertex x are the only possible outgoing edges from $1VOut(u) \setminus C$. Thus, $1VOut(u) \setminus C$ is a 1-vertex-out component of u, which contradicts the minimality of $1VOut(u)$.

We now show that u has two internally vertex-disjoint paths to x in $1VOut(u)$. First, we note that all simple paths from u to x contain only vertices in $1VOut(u)$ since there is no other vertex $x' \neq x$ such that $x' \in 1VOut(u)$ and x' has edges leaving $1VOut(u)$. Assume, for the sake of contradiction, that all paths from u to x in $1VOut(u)$ share a common vertex w. Then, u does not have a path to x in $1VOut(u) \setminus w$. Let C be the set of vertices that become unreachable from u in $1VOut(u) \setminus w$. (Notice that $|C| \geq 1$ since $x \in C$.) Clearly, there is no edge (w', z') such that $w' \in 1VOut(u) \setminus C$ and $z' \in C$, since z' would be reachable from u. Hence, the only vertex that has edges leaving $1VOut(u) \setminus C$ is w. Thus, $1VOut(u) \setminus C$ is a 1-vertex-out component of u, which again contradicts the minimality of $1VOut(u)$. The lemma follows. □

Our algorithm begins with a DFS traversal F_1 from u. We charge to a visited vertex its outgoing edges that were traversed. We stop F_1 when the number of the traversed edges reaches $2\Delta + 1$. Let T be the DFS tree constructed by the DFS traversal. We define the weight of a vertex v, denoted by $w(v)$, to be the total number of edges charged to the descendants of v in T (including v).

Assume that u has a 1-vertex-out component C, with a single vertex x having outgoing edges to $V \setminus C$, containing at most Δ edges. It is easy to see that F_1 is guaranteed to traverse at least $\Delta + 1$ edges outside C (since it visits at least $2\Delta + 1$ edges and $|E(C)| \leq \Delta$), and therefore, since x is the only vertex with outgoing edges from C we have $w(x) \geq \Delta + 1$. Moreover, for any vertex $v \neq u$ whose DFS subtree explores only vertices inside C, we have $w(v) < \Delta$.

Lemma 15. *Let $1VOut(u)$ be a 1-vertex-out component of u such that $|E(1VOut(u))| \leq \Delta$, let x be the only vertex that has edges leaving $1VOut(u)$, and let T be a DFS tree generated by a DFS traversal from u that visited $2\Delta + 1$ edges. Then, for each $v \in T[u, x]$ it holds that $w(v) \geq \Delta + 1$ and for each $v \in 1VOut(u) \setminus T[u, x]$ it holds that $w(v) \leq \Delta$.*

Proof. By the fact that x is the only vertex that has edges leaving $1VOut(u)$ and that $1VOut(u)$ contains at most Δ edges, the only way a DFS traversal can visit $2\Delta + 1$ edges is by visiting at least $\Delta + 1$ edges outside of $1VOut(u)$. It follows that $w(x) \geq \Delta + 1$, and therefore, for each $v \in T[u, x]$ it holds that $w(v) \geq \Delta + 1$. Note that x can be used only once by the DFS traversal, and also that any traversal from u that does not visit vertices $v \notin 1VOut(u)$ cannot be charged by $\Delta + 1$ edges. Therefore, none of the subtraversals from vertices $v \in 1VOut(u) \setminus T[u, x]$ could visit vertices outside $1VOut(u)$, since either v was visited after x, or it could not visit x. In both cases v could not use the outgoing edges of x to visit more that $\Delta + 1$ edges. Thus, for each vertex $v \in 1VOut(u) \setminus T[u, x]$, it holds that $w(v) \leq \Delta$. □

After the traversal F_1, we say that a vertex v is *blocked* if $w(v) \geq \Delta + 1$. Next, we start a second traversal F_2 from u (not necessarily a depth-first search) as follows. The traversal F_2 can visit only the vertex u and vertices that are not blocked. We say that the traversal *reaches* a vertex v whenever it traverses an edge incoming to v; thus F_2 can reach blocked vertices but not visit them and all vertices that are visited are also reached by F_2. Whenever F_2 reaches a blocked vertex v, we unblock all blocked vertices on $T[u, v] \setminus v$. (Notice that v itself is not unblocked.) Assuming that there exists a 1-vertex-out component of u with at most Δ edges, this second traversal F_2 has two main properties: (i) it never unblocks x, and (ii) it reaches all edges and vertices in $1VOut(u)$. Since we are interested only in computing a 1-vertex-out component of u containing at most Δ edges (recall that we assumed in the beginning that u can reach at least $2\Delta + 1$ edges), we terminate F_2 whenever it visits $\Delta + 1$ edges. If the traversal F_2 visits $\Delta + 1$ edges we conclude that there is no 1-vertex-out component of u containing at most Δ edges. Before proving the above claim, we first show the following supporting lemma, which says that the blocked vertices form a path in the DFS tree; we call this path the *heavy path* of F_1.

Procedure 1VertexOut(G, u, Δ)

```
Input: Digraph $G = (V, E)$, a vertex $u$, and an integer $\Delta$
Output: Either a 1-vertex-out component of $u$ with at most $2\Delta$ edges or $\emptyset$; if $\emptyset$ is
        returned, then no 1-vertex-out component of $u$ with at most $\Delta$ edges
        exists
1  Execute DFS $F_1$ from $u$ for up to $2\Delta + 1$ edges
2  Let $S_1$ be the vertices reached by $F_1$
3  if $F_1$ cannot reach $2\Delta + 1$ edges then
4      return $G[S_1]$ as 1-edge-out component of $u$
5  else
6      Block in $G$ vertices on the heavy path of $F_1$
7      Execute a DFS $F_2$ from $u$ on $G$ for up to $\Delta + 1$ edges, whenever a blocked
       vertex $v$ is reached: unblock blocked vertices from $u$ to the predecessor of $v$ in
       $F_1$ and continue the DFS without $v$
8      Let $S_2$ be the vertices reached by $F_2$ (including reached but not unblocked
       vertices)
9      if $F_2$ cannot reach $\Delta + 1$ edges then
10         return $G[S_2]$ as 1-vertex-out comp. of $u$
11     else
12         return $\emptyset$
```

Lemma 16. *Let F be a DFS traversal that visits $2\Delta + 1$ edges and let T be its DFS tree. The vertices v with $w(v) \geq \Delta + 1$ form a path in T.*

Proof. Assume, by contradiction, that the vertices v with $w(v) \geq \Delta + 1$ do not form a path on T. That means, there are two vertices x and y with $w(x), w(y) \geq \Delta + 1$ that do not have an ancestor-descendant relation in T, i.e., $T(x) \cap T(y) = \emptyset$. If we count the edges entering x and y in T, this is a contradiction to the fact that F visited only $2\Delta + 1$ edges. Therefore, the vertices v with $w(v) \geq \Delta + 1$ form a path in T. □

Lemma 17. *Let G be a graph where the vertices v with $w(v) \geq \Delta + 1$ are blocked after the DFS traversal F_1. If there exists a 1-vertex-out component of u containing at most Δ edges, then F_2 traverses at most Δ edges. Moreover, if F_2 traverses at most Δ edges, the subgraph induced by the vertices reached by F_2 (including a reached but not unblocked vertex) defines a 1-vertex-out component of u that contains at most $\Delta + 1$ vertices and at most 2Δ edges.*

Proof. Let us first assume that there exists a 1-vertex-out component $1VOut(u)$ of u that contains at most Δ edges and that all edges leaving $1VOut(u)$ share a common source x. By Lemma 15, x is blocked. The traversal F_2 cannot visit more than Δ edges, since u cannot visit vertices $v \notin 1VOut(u)$ avoiding x, and hence, F_2 cannot unblock x.

Now we show the opposite direction. Assume that F_2 visits at most Δ edges. We will show that there exists a 1-vertex-out component $1VOut(u)$ of u that contains at most Δ vertices and at most 2Δ edges and is induced by the vertices reached by F_2. Clearly, if F_2 unblocks the whole path $P_{blocked}$, then it will visit at least $2\Delta + 1$ edges, since F_1 did so. Hence, there is at least one vertex that remains blocked after the traversal of F_2; let v^* be this vertex. Let C be the set of vertices that were reached by F_2. Then, C has at most one blocked vertex, which is v^*, since whenever two vertices of the path $P_{blocked}$ are reached, reaching the vertex further away from u on $P_{blocked}$ unblocks all the blocked vertices on the tree path from u. Notice that all edges leaving C are from v^*. Moreover, v^* might have at most Δ edges to vertices in C that were not traversed. Thus the subgraph induced by C forms a 1-vertex-out component of u and contains at most 2Δ edges, with the only vertex that has outgoing edges being v^*. Notice that all vertices in C were reached by F_2. We are left to show that there is no 1-vertex-out component $1VOut'(u)$ of u where all the outgoing edges share a common vertex x' and such that $1VOut'(u) \subset 1VOut(u)$. Assume by contradiction that there exists such a component. By Lemma 15 the traversal F_1 would have blocked x', and there is no other outgoing edge from a vertex in $1VOut'(u)$ to a vertex in $1VOut(u) \setminus 1VOut'(u)$. Therefore, F_2 cannot visit vertices outside $1VOut'(u)$ since it cannot unblock x'. A contradiction to the fact that F_2 visited all the vertices of $1VOut(u)$. □

After the execution of the traversal F_2 we can either return a 1-vertex-out component of u with at most 2Δ edges or decide that all 1-vertex-out components of u contain more than Δ edges, as shown in Lemma 6. The pseudocode of our algorithm is illustrated in Procedure 1VertexOut. The following lemma summarizes the result of this section.

Lemma 18. *We compute in $O(\Delta)$ time a 1-vertex-out component of a vertex u containing at most 2Δ edges, or we conclude that there is no 1-vertex-out component of u containing at most Δ edges.*

3.2 Computing the 2-vertex-connected subgraphs.

In this section we present an $O(m\sqrt{m})$ time algorithm for computing the 2-vertex-connected subgraphs of a directed graph. We begin with a simple algorithm and then show how we can improve its running time. Recall that the 2-vertex-connected subgraphs of a graph are subgraphs that do not contain any strong articulation points, that is, they cannot get disconnected by the deletion of any single vertex. In contrast to 2-edge-connected subgraphs, the 2-vertex-connected subgraph do not define a partition of the vertices of the input graph. More specifically, any two 2-vertex-connected subgraphs might share up to one common vertex. This introduces an additional challenge since the existence of a strong articulation point x guarantees that the subsets of two sets of vertices S and $V \setminus (S \cup x)$ do not appear in the same 2-vertex-connected subgraph, but does not provide information on whether x itself appears in a 2-vertex-connected subgraph with vertices from S or $V \setminus (S \cup x)$.

A simple algorithm for computing the 2-vertex-connected subgraphs of a directed graph works as follows. We restrict our attention to the strongly connected components of the input graph. We repeatedly find a strong articulation point x that disconnects the graph into two sets of vertices S and $V \setminus (S \cup x)$, i.e., there is no pair of vertices u and v that are strongly connected in $G \setminus x$ such that $u \in S$ and $v \in V \setminus (S \cup x)$. We recursively execute the same algorithm on the strongly connected components of the subgraphs $G[S \cup x]$ and $G[V \setminus S]$ that contain at least three vertices. If a recursive call fails to identify a strong articulation point in a strongly connected subgraph, then it reports the subgraph as 2-vertex-connected. Both the correctness and the running time of this simple algorithm are easy to verify. First, since at each recursive call we identify a strong articulation point that separates two (non empty) sets of vertices, we know that no pair across these two sets can be in the same 2-vertex-connected subgraph.We moreover restrict the recursive calls on the strongly connected components of the resulting subgraphs since a 2-vertex-connected subgraph is also strongly connected. Therefore, all the 2-vertex-connected subgraphs are preserved at each recursive call, and the algorithm reports a 2-vertex-connected subgraph once it recurses on a subgraph that does not contain a strong articulation point, which is correct by definition. Second, we bound the running time. The maximum recursion depth is $n-1$ since every recursive call is executed on a graph that contains at least one vertex less than the parent call. Although at each recursive call the strong articulation point is included in both sets that it separates, the set of edges is partitioned between the two subgraphs. Therefore, at each recursion level the total number of edges in all instances is at most m, and the total time to compute a strong articulation point and the strongly connected components at the end of each recursive call is $O(m)$, which leads to overall $O(mn)$ running time.

The high-level idea of our algorithm for computing the 2-vertex-connected subgraphs is similar to the algorithm of Section 2.2 for computing the 2-edge-connected subgraphs, but requires some additional machinery. In order to construct the two subgraphs on whose strongly connected components the algorithm recurses we define the following operation. Let G be a digraph, x a vertex, and N a subset of neighbors of x. The operation *split*(x, N) is executed as follows. First, we create an additional vertex x' in G, that serves as a copy of x. Second, for every edge (x, y), where $y \in N$, we remove (x, y) from G and add the edge (x', y). Respectively, for every edge (y, x), where $y \in N$, we remove (y, x) from G and add the edge (y, x'). This operation can be implemented to take time proportional to the number of neighbors of vertices in N, by traversing their edges and for every edge incident to x we change it to be incident to x'.

Lemma 19. *The number of edges in a graph does not change after any split operation. The maximum number of auxiliary vertices after any sequence of split operations is* $2m-n$.

Proof. By definition, no edges are added or deleted while performing the *split* operation. Since every edge has two endpoints, in the worst case all vertices are distinct. Notice that the original n vertices always exist in the graph. Therefore, the total number of auxiliary vertices cannot exceed $2m - n$. □

Lemma 20. *Let G be a directed graph, x a strong articulation point, and N_1, N_2 the neighbors of x such that all paths from any vertex in N_1 to any vertex in N_2 go through*

x. There is a one-to-one correspondence between the 2-vertex-connected subgraphs in G and in G after the execution of either $split(x, N_1)$ or $split(x, N_2)$.

Proof. W.l.o.g., we assume that the *split* operation is $split(x, N_1)$. Let C be a 2-vertex-connected subgraph before the execution of the *split* operation. If the *split* operation is not executed on a vertex of C, then C remains a 2-vertex-connected subgraph. Now assume that the split operation is executed on a vertex $x \in C$. Then all neighbors of x that are in C are strongly connected in $G \setminus x$, and therefore they are all included in N_1 or none of them is. Thus, all the edges between vertices in C are preserved.

Now we prove the opposite direction. Let C be a 2-vertex-connected subgraph after the execution of the *split* operation. Then, either all edges between the vertices in C existed before the *split* operation, or there is an auxiliary vertex $x \in C$ such that all edges between vertices in $C \setminus x$ existed before the operation and all edges between vertices $C \setminus x$ and x were between $C \setminus x$ and a vertex x' before the *split* operation (where x' is the vertex on which the *split* operation was executed). In both cases C was a 2-vertex-connected subgraph before the *split* operation. □

We are ready to describe our algorithm for computing the 2-vertex-connected subgraphs of a directed graph G. We build on the simple recursive algorithm that is described at the beginning of this section. To distinguish the input graph from the graphs in the recursive calls, we refer to the original input graphs as $G_0 = (V_0, E_0)$. We use the terms small components and large components to refer to subgraphs that contains at most and more than $\sqrt{m_0}$ edges, respectively, where $m_0 = |E_0|$. (We allow small components to contain up to $2\sqrt{m_0}$ edges.) Our algorithm begins by identifying all the small 1-vertex-out and 1-vertex-in components of any vertex in G_0, using the algorithm from Section 3.1. Throughout the algorithm we maintain a list L of the vertices for which we should start a search for a small 1-vertex-out or 1-vertex-in component. We show that it is sufficient to search from the vertices that are inserted into L throughout the algorithm in order to find {all} the small 1-vertex-out and 1-vertex-in components of *all* the vertices in the graph. In the initial call to the algorithm we set $L = V_0$. (I.e., this is not done for every recursive call.) At each recursive call the algorithm first tests whether the given strongly connected graph is 2-vertex-connected, and if that's the case, it outputs the graph as a 2-vertex-connected subgraph. Then, while L is not empty, we extract a vertex u from L and search for a small 1-vertex-out or a small 1-vertex-in component of u (containing at most $2\sqrt{m}$ edges).

W.l.o.g., let $1VOut(u)$ be a small 1-vertex-out component of a vertex u that we identify. If all the outgoing edges from $1VOut(u)$ share a common vertex x, then the algorithm executes $split(x, N)$, where N are the neighbors of x in $1VOut(u)$. Furthermore, for every edge $e = (w, z)$ incident to $1VOut(u)$ that is not adjacent to x, insert both w and z into L and remove e from the graph. We treat every identified small 1-vertex-in component in an analogous way.

If, on the other hand, we cannot find new small 1-vertex-out or 1-vertex-in components, we conclude that there are at least two large sets of vertices that are in different strongly connected components, or for every strong articulation point there exist two large sets of vertices that get disconnected by the removal of the strong articulation point. To exploit that, in a final phase of each recursive call we compute the strongly connected components $C_1, C_2, \ldots, C_l$ of the resulting graph after all *split* operations, we execute split $(v, N_{C'})$ on some strong articulation point v from each strongly connected component C_i, where $N_{C'}$ are the neighbors of v that are contained in a singe arbitrary strongly connected component C' in $G[C_i] \setminus v$, and we recursively call the algorithm on each strongly connected component of the resulting graph. Before every recursive call we initialize the lists L to contain the vertices that lost an edge during the last phase of the parent recursive call. We keep this list in order to restrict the total number of searches for small 1-vertex-out and 1-vertex-in components to $O(m+n)$; after initially adding all vertices into the list of the initial call, we only add the endpoints of deleted edges into the lists (which is $O(m)$). Algorithm 2 contains the pseudocode of our algorithm.

Similarly to Algorithm 1 from Section 2.2, we now show the key property that allows us to either find small sets that can be separated by a single vertex deletion or conclude that there are at least two large components that get separated by at most one strong articulation point. Every new 1-vertex-out component that appears in the graph throughout the algorithm must have lost an outgoing edge that is not outgoing from the separating vertex of the component (the only vertex that has outgoing edges from a 1-vertex-out component). Respectively, every new 1-vertex-in component that appears must have lost an incoming edge to a vertex other than the separating vertex of the 1-vertex-in component. Therefore, we use the list L to keep track of the vertices that have lost an edge and for each such vertex u we search for new small 1-vertex-out or 1-vertex-in components of u. If no such small components exist in a set of vertices C, then we know that either (i) C is a 2-vertex-connected subgraph or (ii) we are guaranteed that either two large sets of vertices are in separate strongly connected components of the graph, or that every strong articulation point separates two large sets of vertices. This property is summarized in the following lemma.

Lemma 21. *Let C be a set of vertices in G. Each 1-vertex-out component (of some vertex $u \in C$) in $G[C]$ for which x is the only vertex that has outgoing edges to $V \setminus C$ and that is not a 1-vertex-out component in G must contain an endpoint z of an edge incident to $G[C]$, such that $z \neq x$. Moreover, if there is no 1-vertex-out or 1-vertex-in component containing at most $\sqrt{m}$ edges for any vertex $u \in C$ in both G and $G[C]$, then one of the following holds.*

- *$G[C]$ is a 2-vertex-connected subgraph.*
- *There are two sets $A, B \subset C$ with $|E(G[A])|, |E(G[B])| > \sqrt{m}$ that are disjoint strongly connected components.*
- *For every strong articulation point x, there are two sets $A, B \subset C$ with $|E(G[A])|, |E(G[B])| > \sqrt{m}$ that are separated in $G[C] \setminus x$.*

Proof. We first show that every 1-vertex-out component $1VOut(u)$ of some vertex $u \in C$, where x is the only vertex that has outgoing edges to $V \setminus C$, that is no 1-vertex-out component in G must contain a vertex $w \in 1VOut(u) \setminus \{u, x\}$ such that there is an edge $(w, y) \in G$ with $y \notin C$. Assume, by contradiction, that $1VOut(u)$ exists but there is no such edge (w, y) in G with $w \in 1VOut(u) \setminus \{u, w\}$ and $y \notin C$. In this case, the very same component $1VOut(u)$ is a 1-vertex-out component of u in G, since x is the only vertex having outgoing edges to $V \setminus C$. The same argument on the reverse graph shows that every 1-vertex-in component (of some vertex $u \in C$) in $G[C]$ must contain an endpoind of an edge incident to $G[C]$.

Now we turn to the second part of the lemma. If $G[C]$ is strongly connected and does not contain an articulation point, then $G[C]$ is 2-vertex-connected. If $G[C]$ is not strongly connected, then it contains (at least) two disjoint sets $A, B \subset C$ such that both $G[A]$ and $G[B]$ are strongly connected components of $G[C]$ and $G[A]$ has no outgoing edge in $G[C]$ (i.e., $G[A]$ is a sink in the DAG of SCCs of $G[C]$) and $G[B]$ has no incoming edge in $G[C]$ (i.e., $G[B]$ is a source in the DAG of SCCs of $G[C]$). That is, in $G[C]$ we have that $G[A]$ is a 1-vertex-out component of some $u \in C$ and $G[B]$ is a 1-vertex-in component of some $u' \in C$. Both can have the same property in G or be new such components in $G[C]$ compared to G. In any case it contradicts the assumptions if one of them has at most $\sqrt{m}$ edges and otherwise the lemma holds. If $G[C]$ is strongly connected and contains an articulation point v^*, an analogous argument can be made for two disjoint sets $A, B \subset C$ by considering the DAG of SCCs of $G[C] \setminus v^*$. In this case v^* is the only vertex with incoming edges of B and the only vertex with outgoing edges of A in $G[C]$. Thus the statement of the lemma holds if its assumptions are satisfied. □

Lemma 22. *Algorithm $2VCS$ is correct.*

Proof. First note that by assumption the initial call to the algorithm is on a strongly connected graph and that recursive calls are only made on strongly connected subgraphs. Thus whenever Algorithm $2VCS$ reports a 2-vertex-connected subgraph, then this is a 2-vertex-connected subgraph, since it is strongly connected and does not have any strong articulation points. It suffices to show that $2VCS$ reports all the maximal 2-vertex-connected subgraphs. Notice that this also implies that the reported 2-vertex-connected subgraphs are maximal. Let C be a maximal 2-vertex-connected subgraph. We show that C does not get disconnected by the algorithm, since this will ensure that the algorithm eventually will recurse on C and report it as a 2-vertex-connected subgraph. Since there is no vertex whose deletion separates any pair of vertices in C, any search for either a 1-vertex-out or a 1-vertex-in component of a vertex u, either returns a superset of C, or it fails to identify such a set containing a subset of the vertices of C. Furthermore, note that any deletion of an edge that does not have both endpoints in C does not affect the fact that C is 2-vertex-connected. That is, unless an edge with both endpoints in C is deleted, no strong articulation points appear in C. Thus, it is left to show that no edge (x, y) such that $x, y \in C$ is ever deleted throughout the algorithm. The edges that are deleted are either edges between strongly connected components, two sets of vertices A, B that get disconnected by a strong articulation point, or edges incident to a 1-vertex-out or a 1-vertex-in component found during the course of the algorithm. Since C is always fully included in such a component, no edge of C is deleted. Finally,

Algorithm 2: $2VCS(G, L)$

```
   Input: A strongly connected digraph $G = (V, E)$ and a list of vertices $L$ (initially
          $L = V$)
   Output: The 2-vertex-connected subgraphs of $G$
1  Let $m_0$ be number of edges of initial graph
2  if $|V| \leq 2$ then return $\emptyset$ // removing degenerate subgraphs
3  if $G$ has no strong articulation point then
4      return $\{G\}$ as 2-vertex-connected subgraph
5  while $L \neq \emptyset$ & $G$ has more than $2\sqrt{m_0}$ edges do
6      Extract a vertex $u$ from $L$
7      $S \leftarrow$1VertexOut($G$, $u$, $\sqrt{m_0}$)
8      $S^R \leftarrow$1VertexOut($G^R$, $u$, $\sqrt{m_0}$)
9      Pick non-empty set of $S$ and $S^R$ if it exists
10     Let $x$ be the common vertex in $S$ resp. $S^R$ of all outgoing resp. incoming edges
       (if it exists) and let $N$ be the neighbors of $x$ inside the set
11     Execute $split(x, N)$ (if $x$ exists)
12     Delete all edges incident to the selected set that are not adjacent to $x$ and add
       their endpoints to $L$
13 Compute strongly connected components $C_1, \ldots, C_c$ of $G$
14 $U \leftarrow \emptyset$
15 foreach $C_i, 1 \leq i \leq c$ do
16     Compute a strong articulation point $v$, and execute $split(v, N_{C'})$, where $N_{C'}$
       are the edges between $v$ and the vertices of a single arbitrary strongly
       connected component $C'$ of $C_i \setminus v$.
17     foreach SCC $C$ of $C_i$ do
18         Insert into $L'$ the vertices of $C$ that have incident edge from resp. to
           vertices outside of $C$
19         $U \leftarrow U \cup 2VCS(C, L')$
20 return $U$
```

notice that in each recursive call, unless the graph that is passed to the recursion is 2-vertex-connected, at least one strong articulation point that separates at least one pair of vertices is computed and the algorithm recurses on each strongly connected component (possibly containing a copy of the strong articulation point) after its removal. Thus, finally there will be a recursive call for each strongly connected subgraph that does not contain strong articulation points, including C. □

Lemma 23. *Algorithm $2VCS$ runs in $O(m\sqrt{m})$ time on a graph with m edges.*

Proof. Let $G_0 = (V_0, E_0)$ be the input graph for the initial call to the algorithm. Let $n_0 = |V_0|$ and $m_0 = |E_0|$. First, notice that each time we search for a 1-vertex-out (or a 1-vertex-in component by searching in the reverse graph), we are searching either for a component with at most $\sqrt{m_0}$ edges where all outgoing edges have a common source or for a component with no outgoing edges and at most $2\sqrt{m_0}$ edges. We can identify if such components exist in time $O(\sqrt{m_0})$ by using the algorithm of Section 3.1. We start a search for every vertex that is added to the list L in some recursive call. Notice that initially we add all vertices to L, and throughout the course of the algorithm we insert the two endpoints of every deleted edge into the corresponding list L. The number of edges does not increase by the *split* operations by Lemma 19. Therefore, the total time spent on these calls is $O((m_0 + n_0)\sqrt{m_0}) = O(m_0\sqrt{m_0})$.

Let $G' = (V', E')$ be the graph passed to a recursive call. The algorithm spends $O(|E'|)$ time to test whether there are strong articulation points in the graph (line 3), and additional $O(|E'|)$ time to compute the strong articulation points, execute the *split* operation on an arbitrary strong articulation point in each strongly connected component, and recompute strongly connected components (lines 13–16). Since the recursive calls are executed on subgraphs whose sets of edges are disjoint (since the *split* operator simply partitions the edges incident to the vertex on which the operation is executed, and moreover, all the strongly connected components are disjoint), it follows that the total time spend for the above procedures in all instances at each recursion depth is $O(m_0)$. Notice that the number of vertices does not exceed $2m_0$, by Lemma 19, after any sequence of split operations, and thus this time bound holds for every recursion depth.

Let G' be the graph at some recursive call. We show that the graph passed to each child recursive call has at most $\max\{|E(G')| - \sqrt{m_0}, 2\sqrt{m_0}\}$ edges, or G' is a 2-vertex-connected subgraph and thus the recursion stops. This implies a recursion depth of $O(\sqrt{m_0})$ as follows. If a graph passed to a recursive call has at most $2\sqrt{m_0}$ edges, it means that also the number of vertices is at most $2\sqrt{m_0}$. Therefore, even if the algorithm simply identifies a strong articulation point and executes the *split* operator on it in each recursion and recurses on every strongly connected component of the resulting graph, the total recursion depth is at most $O(\sqrt{m_0})$. On the other hand, there can be at most $\sqrt{m_0}$ cases where the graph that is passed in a recursive call has $\sqrt{m_0}$ fewer edges that G'. Overall, this will prove that the recursion depth is bounded by $O(\sqrt{m_0})$.

It remains to show the claimed bound on the size of the graph passed to a recursive call in line 19. For every 1-vertex-out or 1-vertex-in component S (with less than $2\sqrt{m_0}$ edges) that is discovered throughout the algorithm, the component has either no outgoing (resp., incoming) edges or we execute the *split* operation on the only vertex x that has outgoing (resp., incoming) edges. We can execute the operation *split* in time proportional to the edges incident to the neighbors of x in S, and we can charge this time to the process of identifying the set S (that covers for the edges in $G'[S]$) and to the edges deleted from the graph. By Lemma 20 every 1-vertex-out (resp., 1-vertex-in) component will be in a separate strongly connected component with at most $2\sqrt{m_0}$ edges. Now, let C be the set of vertices that were not included in any 1-vertex-out or any 1-vertex-in component. This set did not contain any 1-vertex-out or any 1-vertex-in component S with less than $\sqrt{m_0}$ edges in G' since otherwise such a set S would contain a vertex x that lost an edge (and thus was added to L) and the algorithm would search for a 1-vertex-out or a 1-vertex-in component of x, identifying S in this way. This means, by Lemma 21, that C either is a 2-vertex-connected subgraph, or there are two disjoint sets in $A, B \subset C$,

$|E'(A)|, |E'(B)| > \sqrt{m_0}$ that are either not strongly connected to each other or separated by at most one strong articulation point in $G'[C]$. If the later holds, A and B will be separated in line 13, and every graph passed to a subsequent recursive call has at most $\max\{|E'(G')| - \sqrt{m_0}, 2\sqrt{m_0}\}$ edges. $\square$

The following theorem summarizes the result of this section.

Theorem 24. *The maximal 2-vertex-connected subgraphs of a digraph can be computed in $O(m^{3/2})$ time.*

4 Extensions

In this section we summarize results that are obtained by extending our algorithms to k-edge-connectivity for $k > 2$. Most of the details are deferred to the full version of [2].

4.1 k-edge-connected subgraphs for digraphs

The high-level idea of the algorithm for computing the 2-edge-connected subgraphs in a digraph easily extends to computing the k-edge-connected subgraphs. However, the algorithm for finding 1-edge-out and 1-edge-in components of a vertex containing at most Δ edges cannot be trivially adjusted to identify $(k-1)$-edge-out and $(k-1)$-edge-in components. We therefore present a new algorithm for computing $(k-1)$-edge-out and $(k-1)$-edge-in components with at most Δ edges. Although the running time of the algorithm is linear in Δ, the dependence on k is exponential:

Lemma 25. *There is an algorithm that, for a given vertex u, computes in time $O((2k)^{k+1} \cdot \Delta)$ a $(k-1)$-edge-out (resp., $(k-1)$-edge-in) component containing u with less than $(2k-1)(\Delta+1)$ edges, or concludes that every $(k-1)$-edge-out (resp., $(k-1)$-edge-in) component of u has more than Δ edges.*

By using Lemma 25 we are able to apply the framework that was used in the algorithm for computing the 2-edge-connected subgraphs to obtain an algorithm for computing the k-edge-connected subgraphs of a digraph. Our result is summarized in the following theorem. The extra $\log n$ factor follows from the fact that an edge cut of size at most $k-1$ can be computed in $O(m \log n)$ time for constant k [8].

Theorem 26. *The maximal k-edge-connected subgraphs of a digraph with m edges and n vertices can be computed in $O(m^{3/2} \log n)$ time for constant k.*

4.1.1 k-edge-connected subgraphs for undirected graphs

The problems of computing the k-edge-connected subgraphs of an undirected graph can be reduced to the equivalent problem for directed graphs in a straightforward way. More specifically, for a given undirected graph we construct a directed graph with the same vertex set, and replace every undirected edge with two bidirectional edges. On the resulting digraph the set of vertices of the k-edge-connected subgraphs are equivalent to the set of vertices of the k-edge-connected subgraphs in the original undirected graph.

The complexity of our algorithms is determined by the choice of the parameter Δ in the algorithm that searches for $(k-1)$-edge-out and the $(k-1)$-edge-in components of a vertex. The parameter Δ determines both the depth of the recursion, which is $O(m/\Delta)$, and the time we spend searching for small components, which is $O((m+n)\Delta)$ in total.

The second factor that affects the complexity is the time spent identifying a cut at every depth of the recursion. Note that the time spent searching for a cut will dominate the $O(m)$ time it takes to compute the strongly connected components before executing the recursive call. This factor is multiplied by the maximum recursion depth in the time complexity of the algorithm. The digraph on which we executed our algorithm originates from an undirected graph, and we can use this to search for edge cuts of size at most $k-1$ faster. Thus, the time complexity of our algorithms is $O(t \cdot (m/\Delta) + n\Delta)$, where t is the time required to identify a cut of size at most $k-1$ in an undirected graph.

The edge cuts of size at most 2 can be identified in linear time [11, 19]. It is easy to verify that the optimal choice of Δ is therefore $m/\sqrt{n}$ for $k = 3$. For constant k, we can compute an edge cut of size at most $(k-1)$ in time $O(m + n \log n)$ [8]. We choose $\Delta = m/\sqrt{n}$ for k-edge-connected subgraphs as well as for 3-edge-connected subgraphs. We obtain the following result.

Theorem 27. *The maximal k-edge-connected subgraphs of an undirected graph can be computed in $O((m + n \log n)\sqrt{n})$ time on a undirected graph with m edges and n vertices. For the maximal 3-edge-connected subgraphs, our algorithm runs in $O(m\sqrt{n})$ time.*

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