Computation of Bivariate Dimension Polynomials of Finitely Generated D-Modules Using Maple

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Abstract

In this paper we present an implementation for computing bivariate dimension polynomials of finitely generated modules over a Weyl algebra in Maple. We recall some basic results in order to explain the notion of dimension polynomials and to introduce methods for their computation based on Gröbner basis techniques. We explain input options for the mentioned implementations and provide several examples.

Keywords:
Weyl algebra, D-module, Berstein polynomial, (x, ∂)-dimension polynomial, (x, ∂)-Gröbner basis

1. Introduction

Bernstein [3] introduced an analog of the Hilbert polynomial for a finitely generated filtered module over a Weyl algebra. Analytical applications of this study can be found, e.g., in Björk's book [5]). In particular, Bernstein [4] was enabled to prove Gelfand's conjecture on meromorphic extensions of functions $\Gamma_f(\lambda) = \int P^\lambda(x) f(x) dx$ in one complex variable λ defined on the half-plane $\mathrm{Re}(\lambda) > 0$ for any polynomial P in n real variables $P(x) = P(x_1, \ldots, x_n)$ and for any function $f(x) = f(x_1, \ldots, x_n) \in C_0^\infty(\mathbb{R}^n)$.

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In [8] the existence of dimension polynomials in two variables associated with the natural bifiltration of a finitely generated module over a Weyl algebra $A_n(K)$ was proved and methods for their computation based on Gröbner basis techniques were given. In what follows we recall algorithms of computation of Bernstein polynomials as well as bivariate dimension polynomials.

The paper is organized as follows. In Section 2 we review some basic concepts of the theory of Weyl algebras and D-modules that are used in the paper. We recall the theorem on Bernstein polynomials of a filtered D-modules. In Section 3 we introduce a bifiltration of a Weyl algebra $A_n(K)$ and define two natural term orderings in $A_n(K)$. Then we define a reduction with respect to two term orderings in a free $A_n(K)$-module giving rise to the definition of (x, ∂)-Gröbner basis. We recall a generalized Buchberger-type algorithm first presented in [8] for their computation and explain how (x, ∂)-Gröbner bases can be used for the computation of bivariate dimension polynomials of finitely generated modules over Weyl algebras. We conclude with some examples of computation of bivariate dimension polynomials.

2. Preliminaries

Throughout the paper $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Q}$ denote the sets of all integers, all non-negative integers and all rational numbers, respectively. $\mathbb{Q}[t]$ denotes the ring of polynomials in one variable t with rational coefficients and $o(t^n)$ denotes a polynomial in $\mathbb{Q}[t]$ of degree less than n. By a ring we always mean an associative ring with unit element. Every ring homomorphism is considered to be unitary (mapping unit element onto unit element), every subring of a ring contains the unit element of the ring. By the module over a ring R we always mean a unitary left R-module.

If for $0 < n \in \mathbb{N}$ we consider an element $a \in \mathbb{N}^n$ then we assume that $a = (a_1, \dots, a_n)$ for $a_1, \dots, a_n \in \mathbb{N}$.

We consider a Weyl algebra as an algebra of differential operators over a polynomial ring. More precisely, let K be a field of zero characteristic, and $0 < n \in \mathbb{N}$. For any set S by $[S]$ we denote the commutative monoid generated by S. We consider indeterminates $x_1, \dots, x_n, \partial_1, \dots, \partial_n$. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring in indeterminates $x_1, \dots, x_n$ and for $i = 1, \dots, n$ consider ∂_i to be the operator on R corresponding to partial differentiation with respect to x_i. Then $A_n(K)$ is defined as the ring of differential operators over R. In other words $A_n(K)$ is obtained by appending $[\partial_1, \dots, \partial_n]$ to the polynomial ring R and equipping $R[\partial_1, \dots, \partial_n]$ with the commutation rules

i. $\partial_i \partial_j = \partial_j \partial_i$, and
ii. $\partial_i r = r \partial_i + \frac{\partial}{\partial x_i}(r)$

for all $i, j \in \{1, \dots, n\}$. A left module over a Weyl algebra is called a *D-module* or $A_n(K)$-module if we want to emphasize n.

Throughout this paper we will use multi-index notation, i.e., for $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ by x^a we denote the term $x_1^{a_1} \cdots x_n^{a_n}$ and by ∂^a we denote the term $\partial_1^{a_1} \cdots \partial_n^{a_n}$. Furthermore by $|a|$ we denote the sum $a_1 + \cdots + a_n$.

The set $\Theta := \{x^a\partial^b \mid a, b \in \mathbb{N}^n\}$ forms a K-basis of $A_n(K)$ (see [5, Chapter 1, Proposition 1.2]). Hence, for every element $D \in A_n(K)$ and $a, b \in \mathbb{N}$ there exist unique coefficients $k_{a,b} \in K$ such that D can be written as a sum $\sum_{a,b\in\mathbb{N}^n} k_{a,b}x^a\partial^b$ with only finitely many $k_{a,b}$ not vanishing. For $D \neq 0$ the number $\operatorname{ord}(D) := \max\{|a| + |b| \mid k_{a,b} \neq 0\}$ is called the order of the element D. We define $\operatorname{ord}(0) := -\infty$.

For $r \in \mathbb{N}$ define sets W_r by

$$W_r := \{D \in A_n(K) \mid \operatorname{ord}(D) \leq r\},$$

and for $0 > r \in \mathbb{Z}$ define $W_r := \{0\}$. Since for any $D_1, D_2 \in A_n(K)\setminus\{0\}$ we have $\operatorname{ord}(D_1D_2) = \operatorname{ord}(D_1) + \operatorname{ord}(D_2)$ the Weyl algebra $A_n(K)$ can be considered as a filtered ring with the nondecreasing filtration $(W_r)_{r\in\mathbb{Z}}$.

Let M be a finitely generated left $A_n(K)$-module with a system of generators $g_1, \dots, g_p$ and for $r \in \mathbb{Z}$ define

$$M_r := \sum_{i=1}^{p} W_r g_i.$$

Then

i. for $r \in \mathbb{Z}$ the set M_r is a finitely generated K-vector space,
ii. for $r, s \in \mathbb{Z}$ we have $W_rM_s = M_{r+s}$, and
iii. $\bigcup_{r\in\mathbb{N}} M_r = M$.

Hence, M can be considered as a filtered $A_n(K)$-module with the filtration $(M_r)_{r\in\mathbb{Z}}$.

The next result is proved in [3] (cf. [5, Chapter 1, Corollaries 3.3, 3.5, and Theorem 4.1]).

Proposition 2.1. *With the above notation, there exists a polynomial $\psi_M(t) \in \mathbb{Q}[t]$ with the following properties:*

i. *$\psi_M(r) = \dim_K(M_r)$ for all sufficiently large $r \in \mathbb{Z}$ (i.e., there exists $r_0 \in \mathbb{Z}$ such that the last equality holds for all integers $r \geq r_0$),*
ii. *$n \leq \deg(\psi(t)) \leq 2n$, and*
iii. *if $a_d, \dots, a_1, a_0 \in \mathbb{Q}$ such that $\psi(t) = a_dt^d + \cdots + a_1t + a_0$, then the degree d of the polynomial $\psi(t)$ and the integer $d!a_d$ do not depend on the choice of the system of generators $g_1, \dots, g_p$ of M. These numbers are denoted by $d(M)$ and $e(M)$, they are called the Bernstein dimension and multiplicity of the module M, respectively.* □

Definition 2.2. *The polynomial $\psi_M(t)$ whose existence is established by Proposition 2.1 is called the* Bernstein polynomial *of the $A_n(K)$-module M associated with the given system of generators. If $d(M) = n$ then M is called* holonomic. *The family of all finitely generated holonomic left $A_n(K)$-modules is called* Bernstein class *and is denoted by $\mathcal{B}_n$.*

Example 2.3. *Let an $A_1(K)$-module M be generated by a single element f satisfying the defining equation*

$$x^a\partial^b f + \partial^{a+b} f = 0 \tag{2.1}$$

with $a, b \in \mathbb{N}\setminus\{0\}$. Then M is isomorphic to the factor module of a free $A_1(K)$-module $A_1(K)e$ with one free generator e by its $A_1(K)$-submodule $N = A_1(K)g$ where $g = (x^a\partial^b + \partial^{a+b})e$. Let π be the natural $A_1(K)$-epimorphism of $A_1(K)e$ onto M, i.e., $\pi : e \mapsto f$, and α the natural $A_1(K)$-epimorphism of the free filtered module F^{a+b} onto the $A_1(K)$-module $N \subseteq M$ equipped with the filtration $(W_r g)_{r\in\mathbb{Z}}$, given by $\alpha : h \mapsto g$. The Bernstein polynomial $\psi_M(t)$ associated with the generator f of the $A_1(K)$-module M can be obtained from the exact sequence of finitely generated filtered modules

$$0 \longrightarrow F^{a+b} \xrightarrow{\alpha} A_1(K)e \xrightarrow{\pi} M \longrightarrow 0$$

where M and $A_1(K)e$ are equipped, respectively, with the filtrations $(W_r e)_{r\in\mathbb{Z}}$ and $(W_r f)_{r\in\mathbb{Z}}$ defined in Section 2, and F^{a+b} is a free filtered $A_1(K)$-module with a single free generator h and filtration $(W_{r-(a+b)}h)_{r\in\mathbb{Z}}$.

Since for all $r \in \mathbb{N}$ sufficiently large we have

$$\begin{aligned}
\dim_K(W_r) &= |\{x^i\partial^j | i + j \leq r\}| \\
&= \binom{r+2}{2}, \text{ and} \\
\dim_K(W_{r-(a+b)}h) &= \binom{r+2-(a+b)}{2}
\end{aligned}$$

for all $r \in \mathbb{Z}$ sufficiently large we obtain

$$\begin{aligned}
\psi_M(r) &= \dim_K(W_r e) - \dim_K(W_{r-(a+b)}h) \\
&= \binom{r+2}{2} - \binom{r+2-(a+b)}{2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\psi_M(t) &= \binom{t+2}{2} - \binom{t+2-(a+b)}{2} \\
&= (a+b)t - \frac{(a+b)(a+b-3)}{2}.
\end{aligned}$$

The following statement (see [5, Chapter 1, Propositions 5.2 and 5.3 as well as Theorem 5.3]) gives some properties of holonomic D-modules.

Proposition 2.4. i. *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of left $A_n(K)$-modules, then $M_2 \in \mathcal{B}_n$ if and only if $M_1 \in \mathcal{B}_n$ and $M_3 \in \mathcal{B}_n$.*

ii. *If $M \in \mathcal{B}_n$, then M has a finite length as a left $A_n(K)$-module. In fact, every strictly increasing sequence of $A_n(K)$-modules contains at most $e(M)$ terms.*

iii. *If M is any filtered $A_n(K)$-module with an increasing filtration $(M_r)_{r\in\mathbb{Z}}$ and there exist positive integers a and b such that $\dim_K(M_r) \leq ar^n + b(r+1)^{n-1}$ for all $r \in \mathbb{N}$, then $M \in \mathcal{B}_n$ and $e(M) \leq n!a$.* □

3. Numerical polynomials in two variables

Definition 3.1. *Let $f(t_1,t_2) \in \mathbb{Q}[t_1,t_2]$ be a polynomial in the two variables t_1 and t_2 with rational coefficients. f is called a* numerical polynomial *if $f(t_1,t_2) \in \mathbb{Z}$ for all $t_1,t_2 \in \mathbb{Z}$ sufficiently large, i.e., there exist $r_0,s_0 \in \mathbb{Z}$ such that $f(r,s) \in \mathbb{Z}$ for all integers $r \geq r_0$ and $s \geq s_0$.*

Obviously, every polynomial $f(t_1,t_2) \in \mathbb{Z}[t_1,t_2]$ is numerical. Now let $0 < m \in \mathbb{N},\ 1 < n \in \mathbb{N}$. Then the polynomial

$$\begin{aligned} g(t_1,t_2) &= \binom{t_1}{m}\binom{t_2}{n} \\ &= \frac{t_1(t_1-1)\cdots(t_1-m+1)}{m!}\cdot\frac{t_2(t_2-1)\cdots(t_2-n+1)}{n!} \end{aligned}$$

with rational coefficients is numerical.

Consider a polynomial $0 \neq f(t_1,t_2) = \sum_{b=(b_1,b_2)\in\mathbb{N}^2} a_b t_1^{b_1} t_2^{b_2} \in \mathbb{Q}[t_1,t_2]$ where only finitely many coefficients $a_b \in \mathbb{Q}$ are not vanishing. By $\deg(f)$, $\deg_{t_1}(f)$ and $\deg_{t_2}(f)$ we denote the total degree, degree with respect to t_1 and degree with respect to t_2 of f, respectively,

$$\begin{aligned} \deg(f) &:= \max\{b_1+b_2 \mid a_b \neq 0\}, \\ \deg_{t_1}(f) &:= \max\{b_1 \mid a_b \neq 0\}, \\ \deg_{t_2}(f) &:= \max\{b_2 \mid a_b \neq 0\}. \end{aligned}$$

The following proposition proved in [13] gives a "canonical" representation of bivariate numerical polynomials we are going to use later.

Proposition 3.2. *Let $f(t_1,t_2) \in \mathbb{Q}[t_1,t_2]$ be a numerical polynomial, and let $deg_{t_1}(f) = p$, $deg_{t_2}(f) = q$. Then for $0 \leq i \leq p$, $0 \leq j \leq q$ there exist uniquelly defined integer coefficients a_{ij} such that*

$$f(t_1,t_2) = \sum_{i=0}^{p}\sum_{j=0}^{q} a_{ij}\binom{t_1+i}{i}\binom{t_2+j}{j}. \tag{3.2}$$

$\square$

Let $m,n \in \mathbb{N}$ and $A \subseteq \mathbb{N}^{m+n}$. Recall that the product order on the set $\mathbb{N}^{m+n}$ is a partial order $\leq_P$ such that $(c_1,\ldots,c_{m+n}) \leq_P (d_1,\ldots,d_{m+n})$ if and only if $c_i \leq d_i$ for all $i = 1,\ldots,m+n$. We define the set V_A by

$$V_A := \{b \in \mathbb{N}^{m+n} \mid \nexists_{a\in A} a \leq_P b\}.$$

For $r,s \in \mathbb{N}$ we define $A(r,s) \subseteq A$ by

$$A(r,s) := \{(a_1,\ldots,a_{m+n}) \in A \mid a_1+\cdots+a_m \leq r,\ a_{m+1}+\cdots+a_{m+n} \leq s\}.$$

The following proposition is a special case of [14, Chapter II, Theorem 2.2.5] and generalizes Kolchin's well-known result on numerical polynomials associated with subsets of $\mathbb{N}^k$ (see [12, Chapter 0, Lemma 17]).

Proposition 3.3. *Let $A \subseteq \mathbb{N}^{m+n}$. Then there exists a numerical polynomial $\omega_A(t_1, t_2)$ in two variables t_1, t_2 such that*

i. $\omega_A(r, s) = |V_A(r, s)|$ *for all sufficiently large* $r, s \in \mathbb{N}$,
ii. $\deg(\omega_A) \leq m + n$, $deg_{t_1}(\omega_A) \leq m$, *and* $deg_{t_2}(\omega_A) \leq n$,
iii. $\deg(\omega_A) = m + n$ *if and only if the set A is empty in which case we have*

$$\omega_A(t_1, t_2) = \binom{t_1 + m}{m}\binom{t_2 + n}{n},$$

and

iv. $\omega_A(t_1, t_2) = 0$ *if and only if* $(0, \ldots, 0) \in A$. □

The following proposition is a special case of [14, Chapter II, Proposition 2.2.11] and provides a formula for the numerical polynomial $\omega_A(t_1, t_2)$ whose existence has been established in Proposition 3.3

Proposition 3.4. *Let $m, n, p \in \mathbb{N}$, $A = \{a_1, \ldots, a_p\}$ a finite subset of $\mathbb{N}^{m+n}$ and for $i = 1, \ldots, p$ let $a_i = (a_{i1}, \ldots, a_{i,m+n})$. Furthermore, for any $l \in \{0, \ldots, p\}$, let $\Gamma(l, p)$ denote the set of all l-element subsets of the set $\mathbb{N}_p = \{1, \ldots, p\}$, and for any $\delta \in \Gamma(l, p)$, $j \in \{1, \ldots, m + n\}$ let*

$$\begin{aligned} \bar{a}_{\delta j} &= \max\{a_{ij} | i \in \delta\} \\ b_\delta &= \sum_{i=1}^{m} \bar{a}_{\delta i}, \text{ and} \\ c_\delta &= \sum_{i=m+1}^{m+n} \bar{a}_{\delta i}. \end{aligned}$$

Then the polynomial $\omega_A(t_1, t_2)$ whose existence has been established by Proposition 3.3 is given by

$$\omega_A(t_1, t_2) = \sum_{l=0}^{p} (-1)^l \sum_{\delta \in \Gamma(l,p)} \binom{t_1 + m - b_\delta}{m}\binom{t_2 + n - c_\delta}{n}.$$

□

4. (x, ∂)-Gröbner bases of submodules in free $A_n(K)$-modules

The use of Gröbner bases for the algorithmic computation of Hilbert polynomials associated with polynomial ideals as well as finitely generated modules over polynomial rings is well understood (see, e.g., [2, Chapter 9] and [10, Section 15.10]). In [11] and [14, Chapter 4] the notion of Gröbner bases has been extended to finitely generated modules over rings of differential operators allowing for the computation of dimension polynomials associated with such modules. In this section we recall the notion of reduction with respect to two orderings and of (x, ∂)-Gröbner bases as introduced in [8].

Let $a, b \in \mathbb{N}^n$ with $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n)$. If $\theta = x^a \partial^b$, then define $\theta_x = x^a := x_1^{a_1} \dots x_n^{a_n}$ and $\theta_\partial := \partial^b = \partial_1^{b_1} \dots \partial_n^{b_n}$. It is easy to see that the sets $\{\theta_x \mid \theta \in \Theta\}$ and $\{\theta_\partial \mid \theta \in \Theta\}$ are commutative multiplicative monoids.

Definition 4.1. *Let $a, b \in \mathbb{N}$ and $\theta = x^a \partial^b \in \Theta$. We define the* x-order $\operatorname{ord}_x(\theta)$ *or* order with respect to $\{x_1, \dots, x_n\}$ *and the* ∂-order $\operatorname{ord}_\partial(\theta)$ *or* order with respect to $\{\partial_1, \dots, \partial_n\}$ *of θ by*

$$\operatorname{ord}_x(\theta) := |a| \qquad \text{and} \qquad \operatorname{ord}_\partial(\theta) := |b|.$$

For all $r, s \in \mathbb{N}$ define the set $\Theta(r, s)$ by

$$\Theta(r, s) := \{\theta \in \Theta \mid \operatorname{ord}_x(\theta) \le r, \ \operatorname{ord}_\partial(\theta) \le s\}.$$

The notions of x-order and ∂-order can be extended to $A_n(K)$ in the following way.

Definition 4.2. *Let $0 \neq D = \sum_{a,b \in \mathbb{N}^n} k_{a,b} x^a \partial^b \in A_n(K)$ where only finitely many $k_{a,b}$ are not vanishing. Then the* x-order $\operatorname{ord}_x(D)$ *and* ∂-order $\operatorname{ord}_\partial(D)$ *of D are defined by*

$$\begin{aligned} \operatorname{ord}_x(D) &= \max\{|a| \mid k_{a,b} \neq 0\}, \text{ and} \\ \operatorname{ord}_\partial(D) &= \max\{|b| \mid k_{a,b} \neq 0\}. \end{aligned}$$

For all $r, s \in \mathbb{N}$ define W_{rs} by

$$W_{rs} := \{D \in A_n(K) \mid \operatorname{ord}_x(D) \le r, \text{ and } \operatorname{ord}_\partial(D) \le s\},$$

and for all $(r, s) \in \mathbb{Z}^2 \setminus \mathbb{N}^2$ let $W_{rs} := 0$. Then we have

i. $W_{rs} \subseteq W_{r+1,s}$ for all $r, s \in \mathbb{Z}$,
ii. $W_{rs} \subseteq W_{r,s+1}$ for all $r, s \in \mathbb{Z}$, and
iii. $\bigcup\{W_{rs} | r, s \in \mathbb{Z}\} = A_n(K)$.

Furthermore, $W_{rs}W_{kl} \subseteq W_{r+k,s+l}$ for any $r, s, k, l \in \mathbb{Z}$ and $W_{rs}W_{kl} = W_{r+k,s+l}$ if $r, s, k, l \in \mathbb{N}$. Hence, we can consider the Weyl algebra $A_n(K)$ as a bifiltered ring with the bifiltration $(W_{rs})_{r,s \in \mathbb{Z}}$.

For $a, b, c, d \in \mathbb{N}^n, \theta, \theta' \in \Theta$ with $\theta = x^a \partial^b$ and $\theta' = x^c \partial^d$ we define two orderings $<_x$ and $<_\partial$ of the set Θ by

$$\begin{aligned} \theta <_x \theta' \quad :\Longleftrightarrow \quad & (\operatorname{ord}_x(\theta), \operatorname{ord}_\partial(\theta), a_1, \dots, a_n, b_1, \dots, b_n) \\ & <_{\text{lex}} (\operatorname{ord}_x(\theta'), \operatorname{ord}_\partial(\theta'), c_1, \dots, c_n, d_1, \dots, d_n), \end{aligned}$$

where $<_{\text{lex}}$ denotes the lexicographic order on $\mathbb{N}^{2n+2}$, and similarly

$$\begin{aligned} \theta <_\partial \theta' \quad :\Longleftrightarrow \quad & (\operatorname{ord}_\partial(\theta), \operatorname{ord}_x(\theta), b_1, \dots, b_n, a_1, \dots, a_n) \\ & <_{\text{lex}} (\operatorname{ord}_\partial(\theta'), \operatorname{ord}_x(\theta'), d_1, \dots, d_n, c_1, \dots, c_n). \end{aligned}$$

Let $\theta = x^a \partial^b$, $\theta' = x^c \partial^d \in \Theta$. We say that θ divides θ' if x^a divides x^c and ∂^b divides ∂^d, that is, $a \le_P c$ and $b \le_P d$ (remember that $\le_P$ denotes the product order on $\mathbb{N}^n$). In this case we also say that θ' is a *multiple* of θ and write $\theta \,|\, \theta'$. Then the monomial $\theta_0 = x^{c-a} \partial^{d-b}$ is denoted by $\frac{\theta'}{\theta}$.

Definition 4.3. *Let* $\theta', \theta'' \in \Theta$. *The* least common multiple $\operatorname{lcm}(\theta', \theta'')$ *of* θ' *and* θ'' *is defined by*

$$\operatorname{lcm}(\theta', \theta'') := \operatorname{lcm}(\theta'_x, \theta''_x) \operatorname{lcm}(\theta'_\partial, \theta''_\partial).$$

Let $A_n(K)E$ be a finitely generated free $A_n(K)$-module with set of free generators $E = \{e_1, \dots, e_q\}$. Then $A_n(K)E$ can be considered as a K-vector space with the basis $\Theta E = \{\theta e_i \mid \theta \in \Theta, 1 \leq i \leq q\}$ whose elements will be called *terms*. For any term θe_j with $\theta \in \Theta, 1 \leq j \leq m$ we define the x-order $\operatorname{ord}_x(\theta e_j)$ and ∂-order $\operatorname{ord}_\partial(\theta e_j)$ of this term by

$$\operatorname{ord}_x(\theta e_j) := \operatorname{ord}_x(\theta), \qquad \text{and} \qquad \operatorname{ord}_\partial(\theta e_j) := \operatorname{ord}_\partial \theta,$$

respectively. If $T \subseteq \Theta$, then let $TE := \{te_i \,|\, t \in T, \, 1 \leq i \leq m\}$. In particular, for any $r, s \in \mathbb{N}$ we have

$$\Theta(r,s)E = \{\theta e_i \mid \operatorname{ord}_x(\theta) \leq r, \ \operatorname{ord}_\partial(\theta) \leq s, \ 1 \leq i \leq m\}.$$

Since the set of all terms ΘE is a basis of the K-vector space $A_n(K)E$, every nonzero element $f \in A_n(K)E$ has a unique representation of the form

$$f = \sum_{\lambda \in \Theta E} a_\lambda \lambda \tag{4.3}$$

where only finitely many a_λ are different from 0. We say that a term λ *appears in* f (or that f *contains* λ) if $a_\lambda \neq 0$.

A term $\lambda = \theta' e_i$ is called a *multiple* of a term $\mu = \theta e_j$ if $i = j$ and $\theta | \theta'$. In this case we also say that μ divides λ, write $\mu | \lambda$ and define

$$\frac{\lambda}{\mu} := \frac{\theta'}{\theta}.$$

We consider two orderings of the set ΘE defined as follows: if $\theta e_i = x^a \partial^b e_i$, $\theta' e_j = x^c \partial^d e_j \in \Theta e$, then

$$\begin{aligned} \theta e_i <_x \theta' e_j \quad &:\Longleftrightarrow \quad (\operatorname{ord}_x(\theta), \operatorname{ord}_\partial(\theta), i, a_1, \dots, a_n, b_1, \dots, b_n) \\ &\qquad\quad <_{\text{lex}} (\operatorname{ord}_x(\theta'), \operatorname{ord}_\partial(\theta'), j, c_1, \dots, c_n, d_1, \dots, d_n), \text{ and} \\ \theta e_i <_\partial \theta' e_j \quad &:\Longleftrightarrow \quad (\operatorname{ord}_\partial(\theta), \operatorname{ord}_x(\theta), i, b_1, \dots, b_n, a_1, \dots, a_n) \\ &\qquad\quad <_{\text{lex}} (\operatorname{ord}_\partial(\theta'), \operatorname{ord}_x(\theta'), j, d_1, \dots, d_n, c_1, \dots, c_n), \end{aligned}$$

where $<_{\text{lex}}$ denotes the lexicographic order.

Definition 4.4. *Let* $0 \neq f = \sum_{\lambda \in \Theta E} a_\lambda \lambda \in A_n(K)E$ *with only finitely many* a_λ *not vanishing. Then the* x-leader $\operatorname{lt}_x(f)$ *and* ∂-leader $\operatorname{lt}_\partial(f)$ *of* f *are defined as the leading terms of* f *with respect to* $<_x$ *and* $<_\partial$*, respectively,*

$$\operatorname{lt}_x(f) := \max_{<_x}\{\lambda \mid a_\lambda \neq 0\}, \qquad \text{and} \qquad \operatorname{lt}_\partial(f) := \max_{<_\partial}\{\lambda \mid a_\lambda \neq 0\}.$$

By $\operatorname{lc}_x(f)$ *and* $\operatorname{lc}_\partial(f)$ *we denote the leading coefficient of* f *with respect to* $<_x$ *and* $<_\partial$*, respectively,*

$$\operatorname{lc}_x(f) := a_{\operatorname{lt}_x(f)}, \qquad \text{and} \qquad \operatorname{lc}_\partial(f) := a_{\operatorname{lt}_\partial(f)}.$$

Now we can formulate the definition of (x, ∂)-Gröbner bases.

Definition 4.5. *Let $E = \{e_1, \ldots, e_q\}$ be a finite set of free generators of a free $A_n(K)$ module $A_N(K)E$ and let N be an $A_n(K)$-submodule of $A_n(K)E$. A finite set $G \subseteq N \setminus \{0\}$ is called an* (x, ∂)-Gröbner basis of N *if for any $0 \neq f \in N$, there exists $g \in G$ such that*

i. $\mathrm{lt}_x(g) | \mathrm{lt}_x(f)$, *and*
ii. $\mathrm{ord}_\partial(\frac{\mathrm{lt}_x(f)}{\mathrm{lt}_x(g)} g) \leq \mathrm{ord}_\partial(f)$.

Remark 4.6. *Let $N \subseteq A_n(K)E$ be a submodule. From condition* i. *of Definition 4.5 of (x, ∂)-Gröbner bases it follows that any (x, ∂)-Gröbner basis of N is also a Gröbner basis of N with respect to $<_x$.*

A finite set of nonzero elements $G \subseteq A_n(K)E$ is said to be an (x, ∂)-Gröbner basis if G is an (x, ∂)-Gröbner basis of the $A_n(K)$-submodule $\sum_{g \in G} A_n(K)g$ it generates.

Definition 4.7. *Let $f, g \in A_n(K)E \setminus \{0\}$ and $h \in A_n(K)E$. If there exists $\theta \in \Theta$ such that*

i. $\mathrm{lt}_x(\theta g) = \mathrm{lt}_x(f)$,
ii. $\mathrm{ord}_\partial(\theta g) \leq \mathrm{ord}_\partial(f)$, *and*
iii. $h = f - \mathrm{lc}_x(f)\theta \frac{g}{\mathrm{lc}_x(g)}$,

then we say that the element f is (x, ∂)-reducible to h modulo g in one step *and write*

$$f \xrightarrow[x,\partial]{g} h.$$

Definition 4.8. *Let $f \in A_n(K)E \setminus \{0\}, h \in A_n(K)E$ and let $G \subseteq A_n(K)E \setminus \{0\}$. If there exist elements $g^{(1)}, g^{(2)}, \ldots, g^{(p)} \in G$ and $h^{(1)}, \ldots, h^{(p-1)} \in E$ such that*

$$f \xrightarrow[x,\partial]{g^{(1)}} h^{(1)} \xrightarrow[x,\partial]{g^{(2)}} \ldots \xrightarrow[x,\partial]{g^{(p-1)}} h^{(p-1)} \xrightarrow[x,\partial]{g^{(p)}} h$$

then we say that f is (x, ∂)-reducible to h modulo G *and write*

$$f \xrightarrow[x,\partial]{G} h.$$

In [8] the following theorem is presented.

Theorem 4.9. *Let $f \in A_n(k)E$ and let $G = \{g_1, \ldots, g_r\} \subseteq A_n(K)E$. Then there exist elements $g \in A_n(K)E$ and $Q_1, \ldots, Q_r \in A_n(K)$ such that*

$$f - g = \sum_{i=1}^{r} Q_i g_i$$

and g is not (x, ∂)-reducible with respect to G.

Algorithm 4.10 `reduction_algorithm`

IN: $f \in A_n(K)E \setminus \{0\}$, $G = \{g_1, \dots, g_r\} \subseteq A_n(K)E \setminus \{0\}$

OUT: An element $g \in A_n(K)E$ such that there exist $Q_1, \dots, Q_r \in A_n(K)$ with $g = f - \sum_{i=1}^{r} Q_i g_i$ and g is not (x,∂)-reducible with respect to G

$Q_1 := 0, \dots, Q_r := 0, g := f$

while there exists $i \in \{1, \dots, r\}$ such that $\operatorname{lt}_x(g_i) | \operatorname{lt}_x(f)$ and $\operatorname{ord}_\partial(\frac{\operatorname{lt}_x(f)}{\operatorname{lt}_x(g_i)} g_i) \leq \operatorname{ord}_\partial(f)$ **do**

  $Q_i := Q_i + \frac{\operatorname{lc}_x(g)}{\operatorname{lc}_x(g_i)} \frac{\operatorname{lt}_x(g)}{\operatorname{lt}_x(g_i)}$

  $g := g - \frac{\operatorname{lc}_x(g)}{\operatorname{lc}_x(g_i)} \frac{\operatorname{lt}_x(g)}{\operatorname{lt}_x(g_i)} g_i$

end while

return g

The process of reduction described in Definition 4.8 can be realized with Algorithm 4.10.

We obtain the following theorem. For a proof we refer to [8].

Theorem 4.11. *Let $G \subseteq A_n(K)E$ be an (x,∂)-Gröbner basis of an $A_n(K)$-submodule N of $A_n(K)E$. Then*

i. $f \in N \setminus \{0\}$ *if and only if* $f \xrightarrow[x,\partial]{G} 0$, *and*

ii. *if $f \in N$ and f is not (x,∂)-reducible with respect to G, then $f = 0$.* □

Definition 4.12. *Let $E = \{e_1, \dots, e_q\}$ be a finite set of free generators of a free $A_n(K)$-module $A_n(K)E$ and let $f, g \in A_n(K)E$. Let*

$$\theta_f^{(x)} = \frac{\operatorname{lcm}(\operatorname{lt}_x(f), \operatorname{lt}_x(g))}{\operatorname{lt}_x(f)}, \qquad \text{and} \qquad \theta_g^{(x)} = \frac{\operatorname{lcm}(\operatorname{lt}_x(g), \operatorname{lt}_x(g))}{\operatorname{lt}_x(g)}.$$

Then the element

$$S_x(f,g) = \frac{1}{\operatorname{lc}_x(f)} \theta_f^{(x)} f - \frac{1}{\operatorname{lc}_x(g)} \theta_g^{(x)} g$$

*is called the x-S-*polynomial *of f and g.*

The following theorem is a generalized version of [1, Lemma 1.7.5.] and has been proved in [8].

Theorem 4.13. *Let $0 < r \in \mathbb{N}$, $f, g_1, \dots, g_r \in A_n(K)E \setminus \{0\}$ and let $\theta_1, \dots, \theta_r \in \Theta, c_1, \dots, c_r \in K$ such that*

$$f = \sum_{i=1}^{r} c_i \theta_i g_i.$$

For all $j, k \in \{1, \dots, r\}$ let $u_{jk} = \operatorname{lcm}(\operatorname{lt}_x(g_j), \operatorname{lt}_x(g_k))$. Suppose that $\theta_1 \operatorname{lt}_x(g_1) = \dots = \theta_r \operatorname{lt}_x(g_r) = u$, $\operatorname{lt}_x(f) <_x u$ and $\operatorname{ord}_\partial(\theta_i g_i) \leq \operatorname{ord}_\partial(f)$ for all $i \in \{1, \dots, r\}$. Then for $1 \leq j, k \leq r$ there exist elements $c_{jk} \in K$ such that

i. $f = \sum_{j=1}^{r}\sum_{k=1}^{r} c_{jk}\theta_{jk}S_x(g_j, g_k)$, *where* $\theta_{jk} := \frac{u}{u_{jk}}$,

ii. *for all* $j, k \in \{1, \ldots, r\}$ *we have* $\theta_{jk}\,\mathrm{lt}_x(S_x(g_j, g_k)) <_x u$, *and*

iii. *for all* $j, k \in \{1, \ldots, r\}$ *we have* $\mathrm{ord}_\partial(\theta_{jk}S_x(g_j, g_k)) \leq \mathrm{ord}_\partial(f)$.

The following result provides the theoretical foundation for the algorithm for constructing (x, ∂)-Gröbner bases. For a proof we refer to [8].

Theorem 4.14. *With the above notation, let* $G = \{g_1, \ldots, g_r\}$ *be a Gröbner basis of an* $A_n(K)$*-submodule* N *of* E *with respect to the order* $<_\partial$*. Furthermore, suppose that for any* $g_i, g_j \in G$ *we have*

$$S_x(g_i, g_j) \xrightarrow[x,\partial]{G} 0.$$

Then G *is an* (x, ∂)*-Gröbner basis of* N.

The last theorem allows one to construct an (x, ∂)-Gröbner basis of an $A_n(K)$-submodule of E starting from a finite Gröbner basis of N with respect to the term order $<_\partial$. The corresponding generalization of Buchberger's algorithm is as follows.

Algorithm 4.15 (x, ∂)-`Gröbner_basis_algorithm`

IN: $\tilde{G} \subseteq E \setminus \{0\}$ a finite Gröbner basis of an $A_n(K)$-submodule N of E with respect to the order $<_\partial$.

OUT: $G \subseteq E \setminus \{0\}$ being an (x, ∂)-Gröbner basis of N.

$G := \tilde{G}$

while there exist $g, g' \in G$ such that $S_x(g, g')$ is not (x, ∂)-reducible to 0 modulo G **do**

$G := G \cup \{\texttt{reduction\_algorithm}(S_x(g, g'), G)\}$

end while

return G

5. Bivariate dimension polynomials associated with $A_n(K)$-modules

Throughout this section we consider the ring $A_n(K)$ as a bifiltered ring with respect to the natural bifiltration $(W_{rs})_{r,s\in\mathbb{Z}}$ introduced above.

Definition 5.1. *Let* M *be a module over a Weyl algebra* $A_n(K)$ *and consider a family* $(M_{rs})_{r,s\in\mathbb{Z}}$ *of* K*-vector subspaces of* M *such that*

i. *if* $r \in \mathbb{Z}$ *is fixed then* $M_{rs} \subseteq M_{r,s+1}$ *for all* $s \in \mathbb{Z}$ *and* $M_{rs} = 0$ *for all sufficiently small* $s \in \mathbb{Z}$*; similarly, if* $s \in \mathbb{Z}$ *is fixed then* $M_{rs} \subseteq M_{r+1,s}$ *for all* $r \in \mathbb{Z}$ *and* $M_{rs} = 0$ *for all sufficiently small* $r \in \mathbb{Z}$,

ii. $\bigcup_{r,s\in\mathbb{Z}} M_{rs} = M$, *and*

iii. *for any* $r, s \in \mathbb{Z}$, $k, l \in \mathbb{N}$ *we have* $W_{kl}M_{rs} \subseteq M_{r+k,s+l}$.

Then $(M_{rs})_{r,s\in\mathbb{Z}}$ *is called a* bifiltration *of* M.

Example 5.2. *Let M be a finitely generated $A_n(K)$-module with generators $f_1, \dots, f_m$ and for $r, s \in \mathbb{Z}$ define the K-vector space M_{rs} by*

$$M_{rs} := \sum_{i=1}^{m} W_{rs} f_i.$$

Then $(M_{rs})_{r,s \in \mathbb{Z}}$ is a bifiltration of the module M which is called a natural bifiltration *of M associated with the system of generators $f_1, \dots, f_m$. Furthermore for any $r, s, k, l \in \mathbb{N}$ we have $W_{kl} M_{rs} = M_{r+k, s+l}$ and the vector space M_{rs} is finitely generated.*

In [8] (x, ∂)-Gröbner bases have been used to prove the existence and obtain a method of computation of bivariate dimension polynomials of finitely generated $A_n(K)$-modules.

Theorem 5.3. *Let M be a finitely generated $A_n(K)$-module with a system of generators $\{f_1, \dots, f_m\}$ and let $(M_{rs})_{r,s \in \mathbb{Z}}$ be the corresponding natural bifiltration of M given for $r, s \in \mathbb{Z}$ by*

$$M_{rs} := \sum_{i=1}^{p} W_{rs} f_i.$$

Then there exists a numerical polynomial $\phi_M(t_1, t_2)$ in two variables t_1, t_2 such that

i. *$\phi_M(r, s) = \dim_K M_{rs}$ for all sufficiently large $(r, s) \in \mathbb{Z}^2$. (That means that there exist $r_0, s_0 \in \mathbb{Z}$ such that the equality holds for all $r \geq r_0$, $s \geq s_0$),*

ii. *$\deg_{t_1}(\phi_M(t_1, t_2)) \leq n$ and $\deg_{t_2}(\phi_M(t_1, t_2)) \leq n$, so that $\deg(\phi_M(t_1, t_2)) \leq 2n$ and for all $0 \leq i, j \leq n$ there exist $a_{ij} \in \mathbb{Z}$ such that the polynomial $\phi_M(t_1, t_2)$ can be represented as*

$$\phi(t_1, t_2) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} \binom{t_1 + i}{i} \binom{t_2 + j}{j}. \tag{5.4}$$

Definition 5.4. *The numerical polynomial $\phi_M(t_1, t_2)$, whose existence is established by Theorem 5.3 is called the* (x, ∂)-dimension polynomial *of the module M associated with the system of generators $\{f_1, \dots, f_m\}$.*

For the computation of the (x, ∂)-dimension polynomial of the module M associated with the system of generators $\{f_1, \dots, f_m\}$ the following theorem is provided in [8].

Theorem 5.5. *Let M be a finitely generated $A_n(K)$-module with system of generators $\{f_1, \dots, f_m\}$, $A_n(K)E$ a free $A_n(K)$-module with basis $E = \{e_1, \dots, e_q\}$, and $\pi : A_n(K)E \longrightarrow M$ the natural $A_n(K)$-epimorphism of $A_n(K)E$ onto M, i.e., $\pi(e_i) = f_i$ for $i = 1, \dots, m$. Furthermore, let the $A_n(K)E$-submodule N be given by $N = \mathrm{Ker}(\pi)$ and let G be an (x, ∂)-Gröbner basis of N. For* *any* $r, s \in \mathbb{N}$, *let* $M_{rs} = \sum_{i=1}^{m} W_{rs} f_i$ *and* $U_{rs} = U'_{rs} \cup U''_{rs}$, *where the sets* $U'_{rs}, U''_{rs} \subseteq \Theta(r,s)E$ *are given by*

$$\begin{aligned} U'_{rs} &= \{\lambda \in \Theta(r,s)E \mid \nexists_{g\in G} \operatorname{lt}_x(g)|\lambda\} \\ U''_{rs} &= \{\lambda \in \Theta(r,s)E \mid \forall_{g\in G,\theta\in\Theta}(\operatorname{lt}_x(\theta g) = \lambda \Longrightarrow \operatorname{ord}_\partial(\theta g) > s)\}. \end{aligned}$$

Then $\pi(U_{rs})$ *is a basis of the* K*-vector space* M_{rs}*. In particular, the* (x,∂)*-dimension polynomial* $\phi_M(t_1,t_2)$ *of the module* M *associated with the system of generators* $\{f_1,\dots,f_m\}$ *for all* $r,s\in\mathbb{N}$ *sufficiently large satisfies*

$$\phi_M(r,s) = |U_{rs}|.$$

Under the conditions of Theorem 5.5 let $\omega_1(t_1,t_2), \omega_2(t_1,t_2) \in \mathbb{Q}[t_1,t_2]$ be two numerical polynomials fulfilling for all $r,s\in\mathbb{N}$ sufficiently large the equations

$$\omega_1(r,s) = |U'_{rs}| \qquad \text{and} \qquad \omega_2(r,s) = |U''_{rs}|.$$

Obviously, $\omega_1(t_1,t_2)$ can be computed using Proposition 3.4. For $\omega_2(t_1,t_2)$ the following approach is provided in [8].

In order to express $|U''_{rs}|$ in terms of r and s, for $1 \le i,j,k,\dots \le d$ let $a_i := \operatorname{ord}_x(\operatorname{lt}_x(g_i))$, $b_i := \operatorname{ord}_\partial(\operatorname{lt}_x(g_i))$, $c_i := \operatorname{ord}_\partial(\operatorname{lt}_\partial(g_i))$, $a_{ij} := \operatorname{ord}_x(\operatorname{lcm}(\operatorname{lt}_x(g_i), \operatorname{lt}_x(g_j)))$, $b_{ij} := \operatorname{ord}_\partial(\operatorname{lcm}(\operatorname{lt}_x(g_i), \operatorname{lt}_x(g_j)))$, $a_{ijk} := \operatorname{ord}_x(\operatorname{lcm}(\operatorname{lt}_x(g_i), \operatorname{lt}_x(g_j), \operatorname{lt}_x(g_k)))$, $b_{ijk} := \operatorname{ord}_\partial(\operatorname{lcm}(\operatorname{lt}_x(g_i), \operatorname{lt}_x(g_j), \operatorname{lt}_x(g_k)))$, ... Then

$$U''_{rs} = \bigcup_{i=1}^{d} \{[\Theta(r-a_i, s-b_i) \setminus \Theta(r-a_i, s-c_i)] \operatorname{lt}_x(g_i)\}.$$

By the combinatorial principle of inclusion and exclusion (see [6, Chapter 5, Theorem 5.1.1]) we obtain

$$\begin{aligned} |U''_{rs}| = & \sum_{i=1}^{d} |\{[\Theta(r-a_i, s-b_i) \setminus \Theta(r-a_i, s-c_i)] \operatorname{lt}_x(g_i)\}| \\ & - \sum_{1\le i<j\le d} |\{[\Theta(r-a_i, s-b_i) \setminus \Theta(r-a_i, s-c_i)] \operatorname{lt}_x(g_i) \\ & \bigcap [\Theta(r-a_j, s-b_j) \setminus \Theta(r-a_j, s-c_j)] \operatorname{lt}_x(g_j)\}| \\ & + \sum_{1\le i<j<k\le d} |\{[\Theta(r-a_i, s-b_i) \setminus \Theta(r-a_i, s-c_i)] \operatorname{lt}_x(g_i) \\ & \bigcap [\Theta(r-a_j, s-b_j) \setminus \Theta(r-a_j, s-c_j)] \operatorname{lt}_x(g_j) \\ & \bigcap [\Theta(r-a_k, s-b_k) \setminus \Theta(r-a_k, s-c_k)] \operatorname{lt}_x(g_k)\}| \\ & - \dots \end{aligned}$$

Furthermore, for any two different elements g_i, g_j, we have

$$\Big| [\Theta(r-a_i, s-b_i) \setminus \Theta(r-a_i, s-c_i)] \operatorname{lt}_x(g_i)$$

$$
\begin{aligned}
&\bigcap[\Theta(r-a_j,s-b_j)\setminus\Theta(r-a_j,s-c_j)]\,\mathrm{lt}_x(g_j)\Big| \\
&= \quad \Big|\Big\{\theta\ \mathrm{lcm}(\mathrm{lt}_x(g_i),\mathrm{lt}_x(g_j)) \mid \theta\in\Theta, \mathrm{ord}_x(\theta)\le r-a_{ij},\ \mathrm{ord}_\partial(\theta)\le s-b_{ij}, \\
&\qquad \mathrm{ord}_\partial\left(\theta\frac{\mathrm{lcm}(\mathrm{lt}_x(g_i),\mathrm{lt}_x(g_j))}{\mathrm{lt}_x(g_i)}\,\mathrm{lt}_\partial(g_i)\right) \\
&\qquad = \mathrm{ord}_\partial(\theta)+b_{ij}-b_i+c_i>s\Big\}\Big| \\
&= \quad |\{\theta \mid \theta\in\Theta, \mathrm{ord}_x(\theta)\le r-a_{ij}, \mathrm{ord}_\partial(\theta)\le s-b_{ij}, \\
&\qquad \mathrm{ord}_\partial(\theta)>s-\min\{c_i+b_{ij}-a_i, c_j+b_{ij}-a_j\}\}| \\
&= \quad \binom{r+n-a_{ij}}{n} \\
&\qquad \left[\binom{s+n-b_{ij}}{n}-\binom{s+n-\min\{c_i+b_{ij}-b_i, c_j+b_{ij}-b_j\}}{n}\right].
\end{aligned}
$$

Similarly, for any three different elements g_i, g_j, g_k we have

$$
\begin{aligned}
&\Big|[\Theta(r-a_i,s-b_i)\setminus\Theta(r-a_i,s-c_i)]\,\mathrm{lt}_x(g_i) \\
&\bigcap[\Theta(r-a_j,s-b_j)\setminus\Theta(r-a_j,s-c_j)]\,\mathrm{lt}_x(g_j) \\
&\bigcap[\Theta(r-a_k,s-b_k)\setminus\Theta(r-a_k,s-c_k)]\,\mathrm{lt}_x(g_k)\Big| \\
&\quad = \binom{r+n-a_{ijk}}{n}\left[\binom{s+n-b_{ijk}}{n}\right. \\
&\qquad \left.-\binom{s+n-\min\{c_i+b_{ijk}-b_i, c_j+b_{ijk}-b_j, c_k+b_{ijk}-b_k\}}{n}\right],
\end{aligned}
$$

and so on.

Thus, for all sufficiently large $(r,s)\in\mathbb{N}^2$, $|U''_{rs}|=\omega_2(r,s)$ where $\omega_2(t_1,t_2)$ is the following numerical polynomial:

$$
\begin{aligned}
\omega_2(t_1,t_2) \quad = \quad & \sum_{i=1}^{d}\binom{t_1+n-a_i}{n}\left[\binom{t_2+n-b_i}{n}-\binom{t_2+n-c_i}{n}\right] \\
& -\sum_{1\le i<j\le d}\binom{t_1+n-a_{ij}}{n}\left[\binom{t_2+n-b_{ij}}{n}\right. \\
& \left.-\binom{t_2+n-\min\{c_i+b_{ij}-b_i, c_j+b_{ij}-b_j\}}{n}\right] \\
& +\sum_{1\le i<j<k\le d}\binom{t_1+n-a_{ijk}}{n}\left[\binom{t_2+n-b_{ijk}}{n}\right. \\
& \left.-\binom{t_2+n-\min\{c_i+b_{ijk}-b_i, c_j+b_{ijk}-b_j, c_k+b_{ijk}-b_k\}}{n}\right]
\end{aligned} \tag{5.5}
$$

$- \dots$

We have implemented the described algorithms and formulas in the Maple$^{\text{TM}}$ package *xd* available at [9] for the case $K = \mathbb{Q}$ which provides an easy way for computing (x, ∂)-dimension polynomials. Our implementation makes utilizes the Maple$^{\text{TM}}$ packages *Ore_Algebra* and *Groebner* for computations in free D-modules, Gröbner basis computations and computations of S-polynomials, respectively. Both packages are part of Chyzak's Mgfun project [7]. In the following examples we will first compute (x, ∂)-dimension polynomials by hand and verify our results using the implementation.

Example 5.6. *With the notation of Theorem 5.3 let $n = 1$ and let an $A_1(K)$-module M be generated by a single element f satisfying the defining equation*

$$x^2 f + \partial^2 f + x\partial f = 0.$$

In other words, M is isomorphic to the factor module of a free $A_1(K)$-module $A_1(K)e$ with a free generator e by its $A_1(K)$-submodule $N = A_1(K)g$ where

$$g = (x^2 + \partial^2 + x\partial)e.$$

Clearly, $\{g\}$ is an (x, ∂)-Gröbner basis of N. Applying Proposition 3.4 (and using the notation of Theorem 5.5), we obtain $\mathrm{lt}_x(g) = x^2 e$, $\mathrm{lt}_\partial(g) = \partial^2 e$, and

$$\begin{aligned} \omega_1(t_1, t_2) &= \binom{t_1+1}{1}\binom{t_2+1}{1} - \binom{t_1+1-2}{1}\binom{t_2+1}{1} \\ &= 2t_2 + 2. \end{aligned}$$

Furthermore, formula (5.5) shows that

$$\begin{aligned} \omega_2(t_1, t_2) &= \binom{t_1+1-2}{1}\left[\binom{t_2+1}{1} - \binom{t_2+1-2}{1}\right] \\ &= 2t_1 - 2. \end{aligned}$$

Thus, the (x, ∂)-dimension polynomial of the module M associated with the generator f is given as

$$\phi_M(t_1, t_2) = \omega_1(t_1, t_2) + \omega_2(t_1, t_2) = 2t_1 + 2t_2.$$

We load the package in Maple^{TM} by

```
> libname := libname, "/path/to/xd.mla":
> with(xd);
```

$$[DimensionPolynomial]$$

The package exports the procedure DimensionPolynomial which accepts a list or set S of elements of $A_n(\mathbb{Q})$ as input and returns the (x, ∂)-dimension polynomial of the module generated by the elements of S associated with the given generators. Since using a standard keyboard layout it is quite troublesome to *input a ∂ symbol, all instances of ∂ are represented by d. In the case of cyclic modules, i.e., generated by one element, specifying the generator is not necessary. If $n = 1$ then x_1 and x as well as d_1 and d are considered to be identical, respectively.*

```
> DimensionPolynomial({x^2 + d^2 + xd});
```

$$2t_1 + 2t_2$$

```
> DimensionPolynomial({(x^2 + d^2 + xd)f});
```

$$2t_1 + 2t_2$$

```
> DimensionPolynomial([x_1^2 + d^2 + xd]);
```

$$2t_1 + 2t_2$$

Hence, we obtain the same result using our implementation.

Example 5.7. *Let M be an $A_2(K)$-module generated by two elements f_1, f_2 satisfying the defining equations*

$$(x_1^3\partial_1^3 + \partial_1^5)f_1 = 0, \qquad \text{and} \qquad x_2^2 f_1 - x_1 f_2 = 0.$$

Then M is isomorphic to the factor module of a free $A_2(K)$-module $E = A_2(K)e_1 + A_2(K)e_2$ with free generators e_1, e_2 by its $A_2(K)$-submodule N, where N is generated by g_1 and g_2 defined by

$$\begin{aligned} g_1 &:= x_1^2\partial_1^3 e_1 + \partial_1^5 e_1, \text{ and} \\ g_2 &:= x_2^2 e_1 - x_1 e_2. \end{aligned}$$

A Gröbner basis of N with respect to the order $<_\partial$ is given by $G := \{g_1, g_2, g_3\}$, where

$$g_3 := x_1\partial_1^5 e_2 + 5\partial_1^4 e_2 + 3x_1^2\partial_1^2 e_2 + x_1^3\partial_1^3 e_2.$$

By Definition 4.12 the x-S-polynomial of g_1 and g_2 is given by

$$S_x(g_1, g_2) = x_1^3\partial_1^3 e_2 + 3x_1^2\partial_1^2 e_2 + x_2^2\partial_1^5 e_1,$$

and is (x, ∂)-reducible modulo g_3 to $x_2^2\partial_1^5 e_1 - x_1\partial_1^5 e_2 - 5\partial_1^4 e_2$ which is in turn (x, ∂)-reducible modulo g_1 to 0. Furthermore we have $S_x(g_1, g_3) = S_x(g_2, g_3) = 0$. So by Definition 4.5, G is an (x, ∂)-Gröbner basis of N. Applying Proposition 3.4 and using the notation of Theorem 5.5 we obtain

$$\omega_1(t_1, t_2) = 4 + 3t_1 + 2t_2^2 + \frac{3}{2}t_1^2 t_2 + \frac{3}{2}t_1 t_2^2 + 6t_1 t_2,$$

and formula (5.5) *shows*

$$\omega_2(t_1, t_2) = t_1^2 t_2 - \frac{5}{2}t_1^2 + t_1 t_2 - \frac{5}{2}t_1 - 4t_2 + 10.$$

Thus, the (x, ∂)-dimension polynomial of the module M associated with the generators f_1, f_2 is given by

$$\phi_M(t_1, t_2) = \frac{5}{2}t_1^2 t_2 + \frac{3}{2}t_1 t_2^2 - \frac{5}{2}t_1^2 + 7t_1 t_2 + 2t_2^2 + \frac{1}{2}t_1 - 4t_2 + 14.$$

Using our implementation we obtain

> $DimensionPolynomial(x_1^2 d_1^3 e_1 + d_1^5 e_1, x_2^2 e_1 - x_1 e_2);$

$$\frac{5}{2}t_1^2 t_2 + \frac{3}{2}t_1 t_2^2 - \frac{5}{2}t_1^2 + 7t_1 t_2 + 2t_2^2 + \frac{1}{2}t_1 - 4t_2 + 14$$

which confirms our computations.

Example 5.8. *Let M be an $A_3(K)$-module generated by one element f satisfying the defining equations*

$$(x_2 + 1)f = 0, \qquad x_1 f = 0, \qquad \text{and} \qquad (\partial_3 - 1)f = 0.$$

Then M is isomorphic to the factor module of a free $A_3(K)$-module $E = A_3(K)e$ with free generator e by its $A_3(k)$-submodule N generated by

$$\begin{aligned} g_1 &:= (x_2 + 1)e, \\ g_2 &:= x_1 e, \text{ and} \\ g_3 &:= (\partial_3 - 1)e. \end{aligned}$$

It can be easily verified that $G := \{g_1, g_2, g_3\}$ is a Gröbner basis of N with respect to $<_\partial$. The x-S-polynomials of g_1 and g_2 as well as g_2 and g_3 are given by $S_x(g_1, g_2) = S_x(g_2, g_3) = x_1 e$ which obviously (x, ∂)-reduces to 0 modulo g_2. The x-S-polynomial of g_1 and g_3 is given by $S_x(g_1, g_3) = (x_2 + \partial_3)e$. It is (x, ∂)-reducible modulo g_1 to $(\partial_3 - 1)e$ which, in turn, obviously is (x, ∂)-reducible to 0 modulo g_3. Hence, G is an (x, ∂)-Gröbner basis of N. Applying Proposition 3.4 and using the notation of Theorem 5.5 we obtain

$$\omega_1(t_1, t_2) = \frac{1}{2}t_1 t_2^2 + \frac{3}{2}t_1 t_2 + \frac{1}{2}t_2^2 + t_1 + \frac{3}{2}t_2 + 1,$$

and $\omega_2(t_1, t_2) = 0$. Thus the (x, ∂)-dimension polynomial of the module M associated with the generator f is given by $\phi_M(t_1, t_2) = \omega(t_1, t_2)$. Again we use our implementation to obtain

> $DimensionPolynomial(\{x_1, d_3 - 1, x_2 + 1\});$

$$1 + t_1 + \frac{3}{2}t_2 + \frac{1}{2}t_2^2 + \frac{1}{2}t_1 t_2^2 + \frac{3}{2}t_1 t_2$$

which confirms our computations.

Working with (x, ∂)-dimension polynomials is justified because they carry additional invariants compared to Bernstein polynomials. The following theorem is provided in [8].

Theorem 5.9. *Let M be a finitely generated $A_n(K)$-module with finite system of generators $\{g_1, \ldots, g_p\}$. For $1 \le i, j \le n$ let $a_{ij} \in \mathbb{Z}$ such that*

$$\phi_M(t_1, t_2) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} \binom{t_1 + i}{i} \binom{t_2 + j}{j}$$

is the (x, ∂)-dimension polynomial associated with the system of generators $\{g_1, \ldots, g_p\}$ of M. Furthermore, let $\Lambda = \{(i, j) \in \mathbb{N}^2 \mid 0 \le i, j \le n$ and $a_{ij} \neq 0\}$, and let $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ be the maximal elements of the set Λ relative to the lexicographic and reverse lexicographic orders on $\mathbb{N}^2$, respectively. Then $d = \deg(\phi_M)$, a_{nn}, μ, ν, the coefficients a_{mn}, a_{μ_1,μ_2} a_{ν_1,ν_2} of the polynomial $\phi_M(t_1, t_2)$, and the coefficients of all terms of $\phi_M(t_1, t_2)$of total degree d do not depend on the finite system of generators of the $A_n(K)$-module M this polynomial is associated with.

For $(W_r)_{r\in\mathbb{Z}}$ as introduced in Section 2 and for all $r \in \mathbb{N}$ we have $W_r \subseteq D_{rr} \subseteq W_{2r}$. Let M be an $A_n(K)$-module with system of generators $\{g_1, \ldots, g_p\}$. By $\psi_M(t)$ and $\phi_M(t_1, t_2)$ we denote the Bernstein polynomial and (x, ∂)-dimension polynomial of M associated with $\{g_1, \ldots, g_p\}$, respectively. Then for all $r \in \mathbb{Z}$ sufficiently large we have $\psi_M(r) \le \phi_M(r, r) \le \psi_M(2r)$ which implies $n \le \deg(\psi_M(t)) = \deg(\phi_M(t_1, t_2)) \le 2n$ and M is a holonomic D-module if and only if $\deg(\phi_M(t_1, t_2)) = n$.

The following example of a finitely generated $A_n(K)$-module was first presented in [8] and shows that an (x, ∂)-dimension polynomial $\phi_M(t_1, t_2)$ can carry more invariants than the Bernstein polynomial $\psi_M(t)$.

Example 5.10. *Let M be as in Example 2.3, i.e., M is the $A_1(K)$-module generated by f satisfying the defining equation*

$$x^a \partial^b f + \partial^{a+b} f = 0 \tag{5.6}$$

with $a, b \in \mathbb{N} \setminus \{0\}$. As we saw, M is isomorphic to the factor module of a free $A_1(K)$-module $A_1(K)e$ with a free generator e by its $A_1(K)$-submodule $N = A_1(K)g$ where $g = (x^a\partial^b + \partial^{a+b})e$. Obviously, $\{g\}$ is an (x, ∂)-Gröbner basis of the module N. Since $\mathrm{lt}_x(g) = x^a\partial^b e$ and $\mathrm{lt}_\partial\, g = \partial^{a+b}e$, we obtain (using the notation of Theorem 5.3) that

$$\begin{aligned} \omega_1(t_1, t_2) &= \binom{t_1 + 1}{1}\binom{t_2 + 1}{1} - \binom{t_1 + 1 - a}{1}\binom{t_2 + 1 - b}{1} \\ &= bt_1 + at_2 + a + b - ab. \end{aligned}$$

Furthermore, formula (5.5) *shows*

$$\begin{aligned} \omega_2(t_1, t_2) &= \binom{t_1 + 1 - a}{1}\left[\binom{t_2 + 1 - b}{1} - \binom{t_2 + 1 - (a + b)}{1}\right] \\ &= at_1 + a(1 - a). \end{aligned}$$

Hence, the (x,∂)-dimension polynomial of the module M associated with the generator f is given by

$$\begin{aligned}\phi_M(t_1,t_2) &= \omega_1(t_1,t_2)+\omega_2(t_1,t_2)\\ &= (a+b)t_1+at_2+2a+b-ab-a^2.\end{aligned}$$

In Example 2.3 it was shown that the Bernstein polynomial $\psi_M(t)$ of M associated with f is given by

$$\begin{aligned}\psi_M(t) &= \binom{t+2}{2}-\binom{t+2-(a+b)}{2}\\ &= (a+b)t-\frac{(a+b)(a+b-3)}{2}.\end{aligned}$$

The Bernstein polynomial carries two invariants, its degree 1 and the leading coefficient $a+b$. The (x,∂)-dimension polynomial carries three invariants, its total degree 1, $a+b$, and a.

Example 5.10 suggests an application of the (x,∂)-dimension polynomial to the isomorphism problem for D-modules. The following example shows that it is possible that two non-isomorphic finitely generated modules over a Weyl algebra have the same set of invariants carried by Bernstein polynomials of the modules, but have different sets of invariants carried by their (x,∂)-dimension polynomials.

Example 5.11. *Consider two cyclic $A_1(K)$-modules M_1 and M_2 generated by m_1 and m_2 satisfying the defining equations*

$$x^4m_1=0, \qquad x^3\partial m_1=0,$$

and

$$x^2\partial^2m_2=0, \qquad x^3\partial m_2=0,$$

respectively. Then M_1 and M_2 are isomorphic to the factor modules of a free $A_1(K)$-module $A_1(K)e$ with free generator e by its $A_1(K)$-submodules N_1 and N_2 generated by $\{x^4e, x^3\partial e\}$ and $\{x^2\partial^2e, x^3\partial e\}$, respectively. As in Example 2.3 we obtain that the Bernstein polynomial associated with M_1 is given by

$$\psi_{M_1}(t)=2t+1,$$

and the Bernstein polynomial associated with M_2 is given by

$$\psi_{M_2}(t)=2t+1.$$

It can be easily verified that $G_1:=\{x^2e\}$ and $G_2:=\{x\partial e\}$ are Gröbner bases of N_1 and N_2, respectively, with respect to to the order $<_\partial$. Since G_1 and G_2 consist of one element each, there are no x-S-polynomials to consider. Hence, G_1 and G_2 are (x,∂)-Gröbner bases of N_1 and N_2, respectively. Applying Proposition

3.4 and using the notation of Theorem 5.5 we obtain that the (x, ∂)-dimension polynomial associated with M_1 is given by

$$\phi_{M_1}(t_1, t_2) = 2t_2 + 2,$$

and the (x, ∂)-dimension polynomial associated with M_2 is given by

$$\phi_{M_2}(t_1, t_2) = t_1 + t_2 + 1.$$

> $DimensionPolynomial(\{x^4 e, x^3 de\});$

$$2t_2 + 2$$

> $DimensionPolynomial(\{x^2 d^2 e, x^3 de\});$

$$t_1 + t_2 + 1$$

It follows from Theorem 5.9 that M_1 and M_2 are not isomorphic as modules over the Weyl algebra $A_1(K)$.

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