Final Report

of

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Legal Disclaimer

Overview

During my time at Michigan State University I made myself familiar with existing literature and state of the art methods in local regularization. Together with Prof. Lamm and under her supervision, I worked on a new method for local regularization (see e.g. [3, 4, 5, 6, 10, 11, 12, 13, 14, 15, 16, 23]) which is expected to be useful for sparse recovery in Inverse Problems (see e.g. [1, 2, 7, 9, 17, 18, 20, 21, 22]). In the (ongoing) scientific cooperation we have already been able to show convergence properties of the regularized solutions corresponding to the proposed new method (see details below), and are currently working on the implementation of the method to gain deeper insights in the computational behaviour and to do further practical analysis.

The next steps will be to finish the implementation of the practical example and analyze the features of the resulting method, to continue the research on the convergence properties of the new local regularization approach and also on the speed of convergence.

Practical Information

While staying at Michigan State Universtiy I was taken great care of by my host, Professor Patricia K Lamm, and I owe her great thanks for that matter. Even though the apartment I rented from the University Housing Departement on campus was basically furnished, it still lacked many essential things for living (e.g. lamps, kitchen equipment, sheets and linen, things for cleaning, office equipment and others more). These things will eventually be needed during such a long stay abroad, but are all together very expensive if they have to be bought just to be left behind in the end.

For my scientific work I was kindly provided office space at the Department of Mathematics of Michigan State University, as well as access to the library, the internet and a printer.

Local Regularization of Integral Equations

Introduction

In the present paper we are concerned with finding approximate solutions of a linear, ill-posed operator equation of the form

$$Au = f, \tag{1}$$

where $A : X \to X$ is an integral operator on a Hilbert space X. We assume to be given noisy data f^δ, satisfying

$$\|f - f^\delta\| \leq \delta.$$

It is well-known that some sort of regularization is needed to deal with these kinds of problems (see e.g. [8, 19]) and one approach which has proven successful, especially for Volterra Integral equations, is local regularization (see e.g. [3, 4, 5, 6, 10, 11, 12, 13, 14, 15, 16, 23]). In the following we will consider a new splitting of a Fredholm Integral operator A in a local and a global part, in good agreement with the principles of local regularization, and prove weak convergence of the resulting regularized solutions to a solution of the original problem (1).

Preliminaries

Definition 1. Let $\Omega = [0,1]^n$ and define the Hilberst space X as

$$X = \{u \in L^2(\mathbb{R}^n) \ : \ \text{supp(u)} \subset \Omega\}.$$

Let $A : X \to X$ be of the form

$$Au(t) = \int_\Omega k(t,s)u(s)ds, \quad \text{a.e.} \quad t \in \Omega, \tag{2}$$

where the kernel $k \in C^1(\Omega \times \Omega, \mathbb{R})$ satisfies

$$0 \leq \underline{k} \leq k(t,s) \leq \bar{k}, \quad \text{a.e. } (t,s) \in \Omega^2 \quad .$$

Throughout this paper we assume that a solution $\bar{u}$ of problem (1) exists and that it belongs to the set

$$\bar{u} \in X_\varepsilon := \{u \in L^2(\mathbb{R}^n) \ : \ \text{supp(u)} \subset [\varepsilon, 1-\varepsilon]^n\}$$

for arbitrary but fixed $\varepsilon > 0$. For $0 < \alpha < \varepsilon$ we write Ω_α, B_α to denote the sets

$$\Omega_\alpha = [\alpha, 1-\alpha] \quad \text{and} \quad B_\alpha = [-\alpha, \alpha]^n.$$

Moreover we denote the set of all solution of (1) by

$$\mathcal{M} = \{u \in X : Au = f\}. \tag{3}$$

In the following we assume the operator A to be monotone, i.e.

$$\langle Au, u\rangle \geq 0 \quad \text{for all} \quad u \in X.$$

Remark 2. Without loss of generality we will assume that $k(t,t) = 1$ for (almost) all $t \in \Omega$. Indeed, if this should not be satisfied, we can consider instead the rescaled problem

$$\tilde{A}\tilde{u}(t) = \int_\Omega \tilde{k}(t,s)\tilde{u}(s)ds = \tilde{f}(t), \quad \text{a.e. } t \in \Omega \quad ,$$

where we define

$$\begin{aligned}\tilde{k}(t,s) &= \frac{k(t,s)}{\sqrt{k(t,t)}\sqrt{k(s,s)}},\\ \tilde{f}(t) &= \frac{f(t)}{\sqrt{k(t,t)}}.\end{aligned}$$

Having solved this problem we can recover the solution of the original problem as

$$u(t) = \tilde{u}(t)\sqrt{k(t,t)}.$$

All these steps are well defined since k is assumed to be strictly bounded away from zero.

Note that the so found rescaled operator $\tilde{A}$ is again monotone, as can be seen from the following calculation.

$$\langle \tilde{A}u, u\rangle = \langle A\tilde{u}, \tilde{u}\rangle \geq 0.$$

Remark 3. It holds that $A \in \mathcal{L}(X)$ with

$$\|Au\|_X^2 = \int_\Omega k^2(t,s)u^2(s)ds \leq \bar{k}^2 \int_\Omega u^2(s)ds = \bar{k}^2 \|u\|_X^2,$$

so that

$$\|A\|_{\mathcal{L}(X)} \leq \bar{k}.$$

Remark 4. Note that every linear operator between Hilbert spaces is hemicontinuous, since the mapping

$$t \mapsto \langle A(u+tv), w\rangle = \langle Au, w\rangle + t\langle Av, w\rangle$$

is clearly continuous for $t \in [0,1]$ for any choice of $u, v, w \in X$.

Thus the linear, monotone operator $A : X \to X$ is also maximal monotone (cf. [24]).

Lemma 5. *The set $\mathcal{M}$ is weakly closed and convex.*

Proof. It is well known that for maximal monotone operators $Au = f$ holds if and only if

$$\langle f - Av, u - v\rangle \geq 0, \qquad \forall v \in X. \tag{4}$$

Let $\mathcal{M} \ni u_k \rightharpoonup u$ then

$$\langle f - Av, u - v\rangle = \lim_{k\to\infty} \langle f - Av, u_k - v\rangle \geq 0.$$

Moreover if $u_1, u_2 \in \mathcal{M}$ and $0 \leq \tau \leq 1$ then

$$\langle f - Av, \tau u_1 + (1-\tau)u_2 - v\rangle = \tau\langle f - Av, u_1 - v\rangle + (1-\tau)\langle f - Av, u_2 - v\rangle \geq 0,$$

which shows that $\tau u_1 + (1-\tau)u_2 \in \mathcal{M}$.

□

Definition 6. For $\alpha \in (0, \varepsilon), u \in X$ and $D \subset B_\alpha$ we define $S_{\alpha,D} : X \to X_\alpha$ through

$$S_{\alpha,D}u(t) := \begin{cases} \frac{1}{|D|}\int_D u(t+s)ds & \text{for } t \in \Omega_\alpha \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Moreover, we introduce the following shorthand notation

$$\begin{aligned} S_\alpha u(t) &:= S_{\alpha,B_\alpha}(t) = \frac{1}{|B_\alpha|}\int_{B_\alpha} u(t+s)ds \\ T_\alpha u(t) &:= S_{\alpha,B_{\alpha^{n+1}}}(t) = \frac{1}{|B_{\alpha^{n+1}}|}\int_{B_{\alpha^{n+1}}} u(t+s)ds. \end{aligned}$$

Remark 7. The operator $T_\alpha u$ in Defintion 6 can be equivalently expressed as follows.

$$\begin{aligned} T_\alpha u(t) &= \frac{1}{|B_{\alpha^{n+1}}|}\int_{B_{\alpha^{n+1}}} u(t+s)ds \\ &= \frac{1}{|B_{\alpha^{n+1}}|}\int_{B_\alpha} u(t+\alpha^n s)(\alpha^n)^n ds \\ &= \frac{1}{|B_\alpha|}\int_{B_\alpha} u(t+\alpha^n s)ds, \end{aligned}$$

where we used

$$\frac{1}{|B_{\alpha^{n+1}}|}(\alpha^n)^n = (2\alpha^{n+1})^{-n}(\alpha^n)^n = (2\alpha)^{-n} = \frac{1}{|B_\alpha|}.$$

Lemma 8. *The adjoint* $S^*_{\alpha,D} : X_\alpha \to X$ *of* $S_{\alpha,D}$ *is given as*

$$S^*_{\alpha,D} v(t) = \frac{1}{|D|} \int_D \hat{v}(t-s) ds, \tag{6}$$

where for any $v \in X_\alpha$ *we define*

$$\hat{v}(t) := \begin{cases} v(t) & t \in \Omega_\alpha \\ 0 & t \in \Omega \backslash \Omega_\alpha. \end{cases} \tag{7}$$

Proof. To compute the adjoint $S^*_{\alpha,D}$ we note that for any $u \in X, v \in X_\alpha$ we have

$$\begin{aligned}
\langle u, S^*_{\alpha,D} v\rangle_X &= \langle S_{\alpha,D} u, v\rangle_{X_\alpha} \\
&= \int_{\Omega_\alpha} S_{\alpha,D} u(t) v(t) dt \\
&= \frac{1}{|D|} \int_{\Omega_\alpha} v(t) \int_D u(t+s) ds dt \\
&= \frac{1}{|D|} \int_{\Omega_\alpha} \int_\Omega v(t)\ \chi_{t+D}(s)\ u(s) ds dt \\
&= \frac{1}{|D|} \int_\Omega u(s) \int_{\Omega_\alpha} \chi_{s-D}(t)\ v(t) dt ds \\
&= \frac{1}{|D|} \int_{\Omega_\alpha} u(s) \int_{s-D} \hat{v}(t) dt ds \\
&= \int_\Omega u(s) \frac{1}{|D|} \int_D \hat{v}(s-r) dr ds \\
&= \langle u, \frac{1}{|D|} \int_D \hat{v}(. - r) dr\rangle,
\end{aligned}$$

where χ_A denotes the characteristic function of $A \subset \mathbb{R}^n$ and we used the identity

$$s \in t + D \iff s - t \in D \iff t \in s - D,$$

which yields

$$\chi_{t+D}(s) = \chi_{s-D}(t).$$

□

Corollary 9. *If the set* $D \subset \mathbb{R}^n$ *satisfies* $-D = D$*, then*

$$S^*_{\alpha,D} v(t) = \frac{1}{|D|} \int_D \hat{v}(t+s) ds,$$

Proof. If $-D = D$ we get using (6)

$$S^*_{\alpha,D} v(t) = \frac{1}{|D|} \int_D v(t-s) ds = \frac{1}{|D|} \int_{-D} v(t+s) ds = S_{\alpha,D} v(t).$$

$\square$

Lemma 10. *If the set $D \subset \mathbb{R}^n$ satisfies $-D = D$, then it holds true that*

$$S_{\alpha,D}u(t) = \frac{1}{|D|}\chi_D * u(t)$$

and

$$\|S_{\alpha,D}u\| \leq \|u\|.$$

Proof. From the definition of $S_{\alpha,D}$ and the fact that $-D = D$ it follows that

$$\begin{aligned}
S_{\alpha,D}u(t) &= \frac{1}{|D|}\int_D u(t+s)ds \\
&= \frac{1}{|D|}\int_{\mathbb{R}^n} \chi_{t+D}(s)u(s)ds \\
&= \frac{1}{|D|}\int_{\mathbb{R}^n} \chi_D(t-s)u(s)ds = \frac{1}{|D|}\chi_D * u(t)
\end{aligned}$$

The estimate on the norm follows from the theorem of Riesz-Thorin for convolution, which yields

$$\|S_{\alpha,D}u\| \leq \frac{1}{|D|}\,\|\chi_D\|_1\,\|u\| = \|u\|.$$

$\square$

Remark 11. Since the domain Ω is bounded, it holds true for $p \leq 2$ that $L^2(\Omega) \subset L^p(\Omega)$ and

$$\|u\|_p \leq \|u\|_2, \qquad u \in L^2(\Omega).$$

Lemma 12. *For $u \in X_\alpha$ it holds that*

$$0 \leq \int_{\Omega_\alpha} |u(t)| - S_\alpha\,|u|\,(t)dt \leq \frac{2n\alpha}{|B_\alpha|}\,\|u\|. \tag{8}$$

Proof. We know

$$\begin{aligned}
\int_{\Omega_\alpha} S_\alpha\,|u|\,(t)dt &= \frac{1}{|B_\alpha|}\int_{\Omega_\alpha}\int_{B_\alpha} |u(t+s)|\,dsdt \\
&= \frac{1}{|B_\alpha|}\int_{B_\alpha}\int_{\Omega_\alpha} |u(t+s)|\,dtds \\
&= \frac{1}{|B_\alpha|}\int_{B_\alpha}\int_{s+\Omega_\alpha} |u(t)|\,dtds \\
&= \frac{1}{|B_\alpha|}\int_{B_\alpha}\int_{\Omega_\alpha} \chi_{s+\Omega_\alpha}(t)\,|u(t)|\,dtds \\
&= \int_{\Omega_\alpha} |u(t)|\,dt - \frac{1}{|B_\alpha|}\int_{B_\alpha}\int_{\Omega_\alpha} (1-\chi_{s+\Omega_\alpha}(t))\,|u(t)|\,dtds
\end{aligned} \tag{9}$$

Thus,

$$
\begin{aligned}
0 \leq \int_{\Omega_\alpha} |u(t)| - S_\alpha \, |u| \,(t) dt &= \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{\Omega_\alpha} (1 - \chi_{s+\Omega_\alpha}(t)) \; |u(t)| \, dtds \\
&= \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{\Omega_\alpha} \chi_{\Omega_\alpha \backslash (s+\Omega_\alpha)}(t) \; |u(t)| \, dtds \\
&= \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{\Omega_\alpha} \chi_{-(\Omega_\alpha - s)\backslash \Omega_\alpha}(s-t) \; |u(t)| \, dtds \\
&\leq \frac{1}{|B_\alpha|} \int_{B_\alpha} \chi_{-\Omega\backslash\Omega_\alpha} * |u| \,(s) ds \\
&\leq \frac{1}{|B_\alpha|} \|\chi_{-\Omega\backslash\Omega_\alpha} * |u| \,\|_1 \leq \frac{1}{|B_\alpha|} \|\chi_{-\Omega\backslash\Omega_\alpha}\|_1 \, \|u\| \\
&= \frac{1}{|B_\alpha|} \, |\Omega \backslash \Omega_\alpha| \; \|u\| \leq \frac{2n\alpha}{|B_\alpha|} \, \|u\|,
\end{aligned}
$$

where we used that

$$
|\Omega\backslash\Omega_\alpha| = \left| \bigcup_{j=1}^{n} [0,1]^{j-1} \times ([0,\alpha] \cup [1-\alpha, 1]) \times [0,1]^{n-j} \right| \leq n \cdot (2\alpha).
$$

Note that we can define the convolution over all of $\mathbb{R}^n \supset B_\alpha$ in the preceding chain of inequalities, i.e.,

$$
f * g \ (s) = \int_{\mathbb{R}^n} f(s-t) g(t) dt.
$$

□

Remark 13. Omitting the absolute values on u in (9) the equalities remain valid and we obtain

$$
\int_{\Omega_\alpha} u(t) - S_\alpha u(t) dt = \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{\Omega_\alpha} (1 - \chi_{s+\Omega_\alpha}(t)) u(t) dtds
$$

and thus

$$
\left| \int_{\Omega_\alpha} u(t) - S_\alpha u(t) dt \right| \leq \int_{\Omega_\alpha} |u(t)| - S_\alpha \, |u| \,(t) dt \leq \frac{2n\alpha}{|B_\alpha|} \, \|u\|.
$$

Lemma 14. *It holds that*

$$
\|T_\alpha A - T_\alpha A T_\alpha^*\|_{\mathcal{L}(X_\alpha)} \leq \alpha n 2^{-n} \; |B_\alpha| \; \|\nabla_2 k\|_\infty
$$

Proof. Making a change of variable $\rho = \alpha^{-n} s$ it follows from the definition of T_α that

$$
T_\alpha u(t) = \frac{1}{|B_\alpha|} \int_{B_\alpha} u(t + \alpha^n \rho) d\rho.
$$

From the respective definitions of T_α, A and Corollary 9 we thus get for $u \in X_\alpha$,

$$\begin{aligned} T_\alpha Au(t) &= \frac{1}{|B_\alpha|}\int_{B_\alpha} Au(t+\alpha^n\rho)d\rho \\ &= \frac{1}{|B_\alpha|}\int_{B_\alpha}\int_{\Omega_\alpha} k(t+\alpha^n\rho,s)u(s)dsd\rho \\ &= \frac{1}{|B_\alpha|}\int_{\Omega_\alpha}\int_{B_\alpha} k(t+\alpha^n\rho,s)d\rho\ u(s)ds \end{aligned} \tag{10}$$

and, using $-B_\alpha = B_\alpha$, we moreover have

$$\begin{aligned} T_\alpha A(T_\alpha^* u)(t) &= \frac{1}{|B_\alpha|}\int_{\Omega}\int_{B_\alpha} k(t+\alpha^n\rho,s)d\rho\ T_\alpha^* u(s)ds \\ &= \frac{1}{|B_\alpha|}\int_{\Omega}\int_{B_\alpha} k(t+\alpha^n\rho,s)d\rho\ \frac{1}{|B_\alpha|}\int_{B_\alpha}\hat{u}(s+\alpha^n\tau)d\tau ds \\ &= \frac{1}{|B_\alpha|}\int_{B_\alpha}\frac{1}{|B_\alpha|}\int_{B_\alpha}\int_{\Omega} k(t+\alpha^n\rho,s)\hat{u}(s-\alpha^n\tau)ds\ d\rho d\tau \\ &= \frac{1}{|B_\alpha|}\int_{B_\alpha}\frac{1}{|B_\alpha|}\int_{B_\alpha}\int_{\Omega-\alpha^n\tau} k(t+\alpha^n\rho,s+\alpha^n\tau)\hat{u}(s)ds\ d\rho d\tau. \end{aligned}$$

Due to $\tau \in B_\alpha$ and $\Omega_\alpha \subset \Omega - \alpha^n B_\alpha$ if follows from $\mathrm{supp}(\hat{\mathrm{u}}) \subset \Omega_\alpha$ that we can write

$$\begin{aligned} T_\alpha A(T_\alpha^* u)(t) &= \frac{1}{|B_\alpha|}\int_{B_\alpha}\frac{1}{|B_\alpha|}\int_{B_\alpha}\int_{\Omega_\alpha} k(t+\alpha^n\rho,s+\alpha^n\tau)u(s)ds\ d\rho d\tau \\ &= \frac{1}{|B_\alpha|^2}\int_{\Omega_\alpha}\int_{B_\alpha}\int_{B_\alpha} k(t+\alpha^n\rho,s+\alpha^n\tau)d\rho d\tau\ u(s)ds. \end{aligned}$$

Using these representations we get

$$\begin{aligned} &|(T_\alpha A - T_\alpha A T_\alpha^* u)(t)| \\ &= \frac{1}{|B_\alpha|}\cdot \\ &\left|\int_{\Omega_\alpha}\int_{B_\alpha}\left[k(t+\alpha^n\rho,s) - \frac{1}{|B_\alpha|}\int_{B_\alpha} k(t+\alpha^n\rho,s+\alpha^n\tau)d\tau\right]d\rho\ u(s)ds\right| \\ &= \frac{1}{|B_\alpha|^2}\left|\int_{\Omega_\alpha}\int_{B_\alpha}\int_{B_\alpha} k(t+\alpha^n\rho,s) - k(t+\alpha^n\rho,s+\alpha^n\tau)d\tau d\rho\ u(s)ds\right| \\ &= \frac{1}{|B_\alpha|^2}\left|\int_{\Omega_\alpha}\int_{B_\alpha}\int_{B_\alpha}\langle\nabla_2 k(t+\alpha^n\rho,\xi),\alpha^n\tau\rangle d\tau d\rho\ u(s)ds\right| \\ &\le \|\nabla_2 k\|_\infty \alpha^n \frac{1}{|B_\alpha|}\int_{B_\alpha}\|\tau\|_1 d\tau \int_{\Omega_\alpha}|u(s)|\,ds \\ &\le \|\nabla_2 k\|_\infty \alpha^n (n\alpha)\int_{\Omega_\alpha}|u(s)|\,ds, \end{aligned}$$

where we wrote

$$\nabla_2 k(t,s) = \left(\frac{\partial k}{\partial s_1}(t,s), \ldots, \frac{\partial k}{\partial s_n}(t,s) \right)^T,$$

and

$$\|\nabla_2 k\|_\infty = \max \left\{ \left| \frac{\partial k}{\partial s_i}(t,s) \right| \; : \; t,s \in \Omega, 1 \le i \le n \right\}.$$

It then holds true that

$$\begin{aligned} \|(T_\alpha A - T_\alpha A T_\alpha^*)u\|^2_{\mathcal{L}(X_\alpha)} &= \int_{\Omega_\alpha} |T_\alpha A - T_\alpha A T_\alpha^* u)(t)|^2 \, dt \\ &\le \|\nabla_2 k\|^2_\infty \alpha^{2(n+1)} n^2 \left| \int_{\Omega_\alpha} |u(s)| \, ds \right|^2 |\Omega_\alpha| \\ &\le \|\nabla_2 k\|^2_\infty \alpha^{2(n+1)} n^2 \, \|u\|^2_{X_\alpha}, \end{aligned}$$

whence the assertion follows, since $|B_\alpha| = (2\alpha)^n$.

□

Definition 15. We define for $\alpha > 0$ the projection $r_\alpha : X \to X_\alpha$ through

$$r_\alpha u(t) = \begin{cases} u(t) & \text{if} \quad t \in \Omega_\alpha \\ 0 & \text{otherwise} \end{cases} . \tag{11}$$

Lemma 16. *It holds that* $\|r_\alpha\|_{\mathcal{L}(X, X_\alpha)} \le 1$ *and that* $\|\mathrm{id} - r_\alpha\|_{\mathcal{L}(X_\alpha)} \to 0$ *as* $\alpha \to 0$.

Proof. The first inequality follows immediately since for $u \in X$

$$\|r_\alpha u\| \le \|u\|.$$

The second part follows since

$$\|u - r_\alpha u\|^2 = \int_{\Omega \setminus \Omega_\alpha} u^2(t) dt = \int_\Omega \chi_{\Omega \setminus \Omega_\alpha}(t) u^2(t) dt.$$

Here we can apply the Dominated Convergence Theorem, since the integrands are clearly dominated by $u^2(t) \in L^1(\Omega)$. We thus obtain

$$\lim_{\alpha \to 0} \|u - r_\alpha u\| = \int_\Omega \lim_{\alpha \to 0} \chi_{\Omega \setminus \Omega_\alpha}(t) u^2(t) dt = 0.$$

□

Lemma 17. *It holds that*

$$\|T_\alpha A - r_\alpha A\|_{\mathcal{L}(X_\alpha)} \le \alpha n 2^{-n} \, |B_\alpha| \, \|\nabla_1 k\|_\infty$$

Proof. Using the representation of $T_\alpha Au(t)$ in (10) we get for $u \in X_\alpha$,

$$\begin{aligned}
|(T_\alpha A - r_\alpha A)u(t)| &= \left| \int_{\Omega_\alpha} \left[\frac{1}{|B_\alpha|} \int_{B_\alpha} k(t+\alpha^n\rho, s)d\rho - k(t,s) \right] u(s)ds \right| \\
&\leq \frac{1}{|B_\alpha|} \left| \int_{\Omega_\alpha} \int_{B_\alpha} k(t+\alpha^n\rho, s) - k(t,s)d\rho \; u(s)ds \right| \\
&\leq \frac{1}{|B_\alpha|} \left| \int_{\Omega_\alpha} \int_{B_\alpha} \langle \nabla_1 k(t,s), \alpha^n \rho \rangle d\rho \; u(s)ds \right| \\
&\leq \|\nabla_1 k\|_\infty \alpha^n \frac{1}{|B_\alpha|} \int_{B_\alpha} \|\rho\|_1 d\rho \int_{\Omega_\alpha} |u(s)| \, ds \\
&\leq \|\nabla_1 k\|_\infty \alpha^n (n\alpha) \int_{\Omega_\alpha} |u(s)| \, ds.
\end{aligned}$$

It then holds true that

$$\|(T_\alpha A - r_\alpha A)u\|^2_{\mathcal{L}(X_\alpha)} \leq \|\nabla_1 k\|^2_\infty \alpha^{2(n+1)} n^2 \|u\|^2_{X_\alpha},$$

whence the assertion follows, since $|B_\alpha| = (2\alpha)^n$.

□

Local Regularization

With the preliminary analysis from the previous section we are now in position to define the localized approach to regularization which is of main interest in this paper.

Definition 18. We define the following splitting of the operator $T_\alpha A : X_\alpha \to X_\alpha$,

$$T_\alpha Au(t) = \frac{1}{|B_\alpha|} \int_{\Omega_\alpha} \int_{B_\alpha} k(t+\alpha^n\rho, s)d\rho \; u(s)ds = (T_\alpha A)_g u(t) + (T_\alpha A)_\ell u(t), \tag{12}$$

where for $u \in X_\alpha$ we define

$$\begin{aligned}
(T_\alpha A)_g u(t) &:= \frac{1}{|B_\alpha|} \int_{\Omega_\alpha \setminus (t+B_\alpha)} \int_{B_\alpha} k(t+\alpha^n\rho, s)d\rho \; u(s)ds \\
(T_\alpha A)_\ell u(t) &:= \frac{1}{|B_\alpha|} \int_{t+B_\alpha} \int_{B_\alpha} k(t+\alpha^n\rho, s)d\rho \; u(s)ds \\
&= \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{B_\alpha} k(t+\alpha^n\rho, t+s)d\rho \; u(t+s)ds
\end{aligned}$$

Lemma 19. *Let*

$$\kappa_1 = n\varepsilon^n \|\nabla_1 k\|_\infty + n \|\nabla_2 k\|_\infty, \tag{13}$$

then for all $\kappa \geq \kappa_1$ and $u \in X_\alpha$ it holds that

$$\|(T_\alpha A)_\ell u\|_{X_\alpha} \leq |B_\alpha| (1+\kappa\alpha) \|u\|_{X_\alpha}. \tag{14}$$

Proof. Using $k(t,t) = 1$ for all $t \in \Omega$,

$$
\begin{aligned}
(T_\alpha A)_\ell u(t) &= \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{B_\alpha} k(t+\alpha^n \rho, t+s) - k(t, t+s) d\rho \; u(t+s) ds \\
&\quad + \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{B_\alpha} k(t, t+s) - k(t,t) d\rho \; u(t+s) ds + \int_{B_\alpha} u(t+s) ds.
\end{aligned}
$$

We estimate the resulting terms individually as follows. For the first term we obtain

$$
\begin{aligned}
&\frac{1}{|B_\alpha|} \left| \int_{B_\alpha} \int_{B_\alpha} k(t+\alpha^n \rho, t+s) - k(t,t+s) d\rho \; u(t+s) ds \right| \\
&\qquad \leq \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{B_\alpha} |\langle \nabla_1 k(\xi, t+s), \alpha^n \rho \rangle| \, d\rho \quad |u(t+s)| \, ds \\
&\qquad \leq \alpha^n \|\nabla_1 k\|_\infty \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{B_\alpha} \|\rho\|_1 d\rho \quad |u(t+s)| \, ds \\
&\qquad \leq \alpha^n \|\nabla_1 k\|_\infty \; |B_\alpha| \, (n\alpha) \frac{1}{|B_\alpha|} \int_{B_\alpha} |u(t+s)| \, ds \\
&\qquad = n\alpha^{n+1} \|\nabla_1 k\|_\infty \; |B_\alpha| \, S_\alpha(\,|u|)(t),
\end{aligned}
$$

where we wrote

$$
\nabla_1 k(t,s) = \left(\frac{\partial k}{\partial t_1}(t,s), \ldots, \frac{\partial k}{\partial t_n}(t,s) \right)^T,
$$

and

$$
\|\nabla_1 k\|_\infty = \max \left\{ \left| \frac{\partial k}{\partial t_i}(t,s) \right| \; : \; t, s \in \Omega, 1 \leq i \leq n \right\}.
$$

Similarly, we get for the second term

$$
\begin{aligned}
&\frac{1}{|B_\alpha|} \left| \int_{B_\alpha} \int_{B_\alpha} k(t,t+s) - k(t,t) d\rho \; u(t+s) ds \right| \\
&\qquad \leq \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{B_\alpha} |\langle \nabla_2 k(t,\xi), s \rangle| \, d\rho \quad |u(t+s)| \, ds \\
&\qquad \leq \; \|\nabla_2 k\|_\infty \int_{B_\alpha} \|s\|_1 \; |u(t+s)| \, ds \\
&\qquad \leq n\alpha \, \|\nabla_1 k\|_\infty \; |B_\alpha| \, S_\alpha(\,|u|)(t).
\end{aligned}
$$

Thus, we obtain

$$
|(T_\alpha A)_\ell u(t)| \leq \left(n\alpha^{n+1} \|\nabla_1 k\|_\infty + n\alpha \, \|\nabla_2 k\|_\infty + 1 \right) \; |B_\alpha| \, S_\alpha(\,|u|)(t),
$$

whence it follows that

$$
\begin{aligned}
\|(T_\alpha A)_\ell u\|_{X_\alpha} &\leq \left(n\alpha^{n+1} \|\nabla_1 k\|_\infty + n\alpha \, \|\nabla_2 k\|_\infty + 1 \right) \; |B_\alpha| \; \|S_\alpha(\,|u|)\|_{X_\alpha} \\
&\leq (\kappa\alpha + 1) \; |B_\alpha| \; \|u\|_{X_\alpha}.
\end{aligned}
$$

□

Lemma 20. *Let*

$$\kappa_2 = n\varepsilon^n \|\nabla_1 k\|_\infty + 3n \|\nabla_2 k\|_\infty, \tag{15}$$

then for all $\kappa \geq \kappa_2$ and $u \in X_\alpha$ it holds that

$$\langle (T_\alpha A)_\ell u, u\rangle_{X_\alpha} = |B_\alpha| \, \|u\|^2 + \langle G_\alpha u, u\rangle_{X_\alpha}, \tag{16}$$

where

$$-(2+\alpha\kappa) \, |B_\alpha| \, \|u\|^2_{X_\alpha} \leq \langle G_\alpha u, u\rangle_{X_\alpha} \leq \kappa\alpha \, |B_\alpha| \, \|u\|^2_{X_\alpha}.$$

Proof. Using $k(t,t) = 1$ for all $t \in \Omega$ (cf. Remark 2), we find that

$$(T_\alpha A)_\ell u(t) = G_\alpha u(t) + |B_\alpha| \, u(t),$$

where

$$\begin{aligned} G_\alpha u(t) = \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{B_\alpha} k(t + \alpha^n \rho, t+s) - k(t, t+s) d\rho \; u(t+s) ds \\ H_\alpha u(t) + \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{B_\alpha} k(t, t+s) - k(t,t) d\rho \; u(t) ds, \end{aligned}$$

and H_α is defined as

$$H_\alpha u(t) = \frac{1}{|B_\alpha|} \int_{B_\alpha} \int_{B_\alpha} k(t, t+s) d\rho \; (u(t+s) - u(t)) ds.$$

This readily establishes the decomposition (16) and it remains to estimate $\langle G_\alpha u, u\rangle$. To this end consider the terms of $\langle G_\alpha u, u\rangle$ individually. For the contribution of the first term it holds that

$$\begin{aligned} &\frac{1}{|B_\alpha|} \int_{\Omega_\alpha} \int_{B_\alpha} \int_{B_\alpha} k(t + \alpha^n \rho, t+s) - k(t, t+s) d\rho \; u(t+s) ds \; u(t) dt \\ &\quad = \frac{1}{|B_\alpha|} \int_{\Omega_\alpha} \int_{B_\alpha} \int_{B_\alpha} \langle \nabla_1 k(\xi, t+s), \alpha^n \rho \rangle d\rho \; u(t+s) ds \; u(t) dt \\ &\quad \leq \|\nabla_1 k\|_\infty \alpha^n \frac{1}{|B_\alpha|} \int_{B_\alpha} \|\rho\|_1 d\rho \int_{\Omega_\alpha} \int_{B_\alpha} |u(t+s)| \, ds \, |u(t)| \, dt \\ &\quad \leq \|\nabla_1 k\|_\infty \alpha^n (n\alpha) \, |B_\alpha| \, \langle S_\alpha(|u|), |u| \rangle \\ &\quad \leq n\alpha^{n+1} \|\nabla_1 k\|_\infty \, |B_\alpha| \, \|u\|^2, \end{aligned}$$

and, similarly, for the contribution of the last term we obtain

$$\begin{aligned} &\frac{1}{|B_\alpha|} \int_{\Omega_\alpha} \int_{B_\alpha} \int_{B_\alpha} k(t, t+s) - k(t,t) d\rho \; u(t) ds \; u(t) dt \\ &\quad = \frac{1}{|B_\alpha|} \int_{\Omega_\alpha} \int_{B_\alpha} \int_{B_\alpha} \langle \nabla_2 k(t, \xi), s \rangle d\rho ds \; u^2(t) dt \\ &\quad \leq \|\nabla_2 k\|_\infty \int_{B_\alpha} \|s\|_1 ds \, \|u\|^2 \\ &\quad \leq n\alpha \|\nabla_2 k\|_\infty \, |B_\alpha| \, \|u\|^2. \end{aligned}$$

Altogether this shows that

$$\langle G_\alpha u, u\rangle \le \langle H_\alpha u, u\rangle + \kappa_1 \alpha \; |B_\alpha| \; \|u\|^2.$$

To give an estimate for $\langle H_\alpha u, u\rangle$ we write

$$\begin{aligned}
\langle H_\alpha u, u\rangle \le & \int_{\Omega_\alpha}\int_{B_\alpha} |k(t,t+s)-k(t,t)| \cdot |u(t+s)-u(t)| \, ds \; |u(t)| \, dt \\
& + \int_{\Omega_\alpha}\int_{B_\alpha} u(t+s)-u(t) ds \; u(t) dt \\
& \le n\alpha \, \|\nabla_2 k\|_\infty \; |B_\alpha| \left(\langle S_\alpha \, |u| \, , |u|\rangle + \|u\|^2\right) + |B_\alpha| \left(\langle S_\alpha u, u\rangle - \|u\|^2\right) \qquad (17) \\
& \le 2n\alpha \, \|\nabla_2 k\|_\infty \; |B_\alpha| \; \|u\|^2.
\end{aligned}$$

We thus obtain the upper bound

$$\langle G_\alpha u, u\rangle \le \kappa_2 \alpha \; |B_\alpha| \; \|u\|^2.$$

Inserting absolut values into (17) and using Lemma 10 it holds

$$|\langle H_\alpha u, u\rangle| \le 2n\alpha \, \|\nabla_2 k\|_\infty \; |B_\alpha| \; \|u\|^2 + 2 \; |B_\alpha| \; \|u\|^2,$$

which establishes the lower bound through

$$|\langle G_\alpha u, u\rangle| \le (2 + \alpha\kappa_2) \; |B_\alpha| \; \|u\|^2.$$

□

The following corollary is an immediate consequence of the previous Lemma.

Corollary 21. *It holds true that*

$$|\langle (T_\alpha A)_\ell u, u\rangle| \le \; |B_\alpha| \, (1 + \alpha\kappa_2) \, \|u\|^2. \tag{18}$$

Let us now define what will be the regularizing approximation of the local part of $T_\alpha A$.

Definition 22. Let $\nu > 1$ and $c > \kappa_1$ be fixed. We define

$$a_\alpha := (\nu + c\alpha) \; |B_\alpha| \tag{19}$$

Lemma 23. *With κ_1 as in* (13) *it holds that*

$$\|(T_\alpha A)_\ell - a_\alpha \mathrm{id}\|_{\mathcal{L}} \le (1 + \nu + (\kappa_1 + c)\alpha) \; |B_\alpha| \, . \tag{20}$$

Proof. Using Lemma 19 with $\kappa = \kappa_1$, we obtain

$$\begin{aligned}
\|(T_\alpha A)_\ell u - a_\alpha u\|^2 = & \; \|(T_\alpha A)_\ell u\|^2 - 2\langle (T_\alpha A)_\ell u, a_\alpha u\rangle + a_\alpha^2 \, \|u\|^2 \\
& \le \left((1+\kappa_1\alpha)^2 + 2(1+\kappa_1\alpha)(\nu + c\alpha) + (\nu + c\alpha)^2\right) \; |B_\alpha|^2 \; \|u\|^2 \\
& = (1 + \nu + (\kappa_1 + c)\alpha)^2 \; |B_\alpha|^2 \; \|u\|^2,
\end{aligned}$$

whence the assertion follows.

$\square$

Lemma 24. *With* κ_1 *as in* (13) *it holds that*

$$\langle \big(a_\alpha \mathrm{id} - (T_\alpha A)_\ell\big)u, u\rangle \geq (\nu - 1 + (c-\kappa_1)\alpha)\ |B_\alpha|\ \|u\|^2. \tag{21}$$

Proof. The estimate is an immediate consequence of (14) and (19).

$\square$

Lemma 25. *Let*

$$\kappa_3 = n\varepsilon^n \|\nabla_1 k\|_\infty + (1+2^{-n})n\|\nabla_2 k\|_\infty, \tag{22}$$

then it holds that

$$\langle \big((T_\alpha A)_g + a_\alpha \mathrm{id}\big)u, u\rangle \geq (\nu - 1 + (c-\kappa_3)\alpha)\ |B_\alpha|\ \|u\|^2. \tag{23}$$

Proof. Using Lemma 14 and 24, we obtain

$$\begin{aligned}
&\langle \big((T_\alpha A)_g + a_\alpha \mathrm{id}\big)u, u\rangle \\
&\quad = \langle \big(T_\alpha A - T_\alpha A T_\alpha^*\big)u, u\rangle + \langle T_\alpha A T_\alpha^* u, u\rangle + \langle \big(a_\alpha \mathrm{id} - (T_\alpha A)_\ell\big)u, u\rangle \\
&\quad \geq -\alpha n 2^{-n}\ |B_\alpha|\ \|\nabla_2 k\|_\infty \|u\|^2 + (\nu - 1 + (c-\kappa_1)\alpha)\ |B_\alpha|\ \|u\|^2 \\
&\quad = (\nu - 1 + (c-\kappa_3)\alpha)\ |B_\alpha|\ \|u\|^2.
\end{aligned}$$

whence the assertion follows.

$\square$

Lemma 26. *For* $\alpha > 0$ *small enough the operator*

$$\big((T_\alpha A)_g + a_\alpha \mathrm{id}\big)^{-1} : X_\alpha \to X_\alpha$$

exists and it holds that

$$\|\big((T_\alpha A)_g + a_\alpha \mathrm{id}\big)^{-1}\|_{\mathcal{L}} \leq \frac{1}{c(\alpha)}, \tag{24}$$

where

$$c(\alpha) = (\nu - 1 + (c-\kappa_3)\alpha)\ |B_\alpha|\,. \tag{25}$$

Proof. From Lemma 25 we see that $\big((T_\alpha A)_g + a_\alpha \mathrm{id}\big) : X_\alpha \to X_\alpha$ is strongly monotone (for α small enough). Moreover, every linear operator is clearly hemicontinuous so that it follows from [24, Proposition 32.7] that $\big((T_\alpha A)_g + a_\alpha \mathrm{id}\big)$ is maximal monotone. From the strong monotonicity we also obtain injectivity and coercivity with respect to 0 and using [24, Proposition 32.27] we see that

$$R\big((T_\alpha A)_g + a_\alpha \mathrm{id}\big) = X_\alpha.$$

We have thus shown that $\big((T_\alpha A)_g + a_\alpha \mathrm{id}\big)^{-1}$ exists and is single valued. To complete the proof we use Lemma 25 to obtain

$$\begin{aligned}(\nu - 1 + (c-\kappa_3)\alpha)\ |B_\alpha|\ &\|\big((T_\alpha A)_g + a_\alpha \mathrm{id}\big)^{-1} u\|^2 \\ &\le \langle u, \big((T_\alpha A)_g + a_\alpha \mathrm{id}\big)^{-1} u\rangle \\ &\le\ \|u\|\,\|\big((T_\alpha A)_g + a_\alpha \mathrm{id}\big)^{-1} u\|,\end{aligned}$$

whence the estimate in the assertion follows.

□

Convergence

Definition 27. Let $\alpha > 0$, then we choose the regularized solution $u_\alpha \in X_\alpha$ such that

$$(T_\alpha A)_g u_\alpha + a_\alpha u_\alpha = T_\alpha f \tag{26}$$

Lemma 28. *It holds that*

$$u_\alpha - \bar{u} = \big((T_\alpha A)_g + a_\alpha \mathrm{id}\big)^{-1} ((T_\alpha A)_\ell - a_\alpha \mathrm{id})\bar{u}.$$

Proof. We use

$$((T_\alpha A)_g + (T_\alpha A)_\ell)\bar{u} = T_\alpha f \tag{27}$$

□

We will now see that the regularized solutions from Definition 27 converge weakly to the set $\mathcal{M}$ of solutions of $Au = f$. To this end we will need the following Lemma.

Lemma 29. *Assume that a family* $\{u_\alpha\}$ *satisfying* (26) *is bounded and converges weakly to* $u \in X$*, then for all* $v \in X$

$$\langle T_\alpha A(u_\alpha - r_\alpha v), u_\alpha - r_\alpha v\rangle \to \langle f - Av, u - v\rangle \quad \textit{as} \quad \alpha \to 0.$$

Proof. By the virtue of (12) and (26) we obtain

$$\begin{aligned}\langle T_\alpha A(u_\alpha - r_\alpha v), u_\alpha - r_\alpha v\rangle &= \langle \big((T_\alpha A)_\ell + (T_\alpha A)_g\big) u_\alpha - T_\alpha A r_\alpha v, u_\alpha - r_\alpha v\rangle \\ &= \langle ((T_\alpha A)_\ell - a_\alpha) u_\alpha + T_\alpha f - T_\alpha A r_\alpha v, u_\alpha - r_\alpha v\rangle \\ &= \sum_{i=1}^{5} T_i,\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \langle ((T_\alpha A)_\ell - a_\alpha) u_\alpha, u_\alpha - r_\alpha v\rangle \\
T_2 &= \langle T_\alpha f - r_\alpha f, u_\alpha - r_\alpha v\rangle \\
T_3 &= \langle r_\alpha f - r_\alpha A v, u_\alpha - r_\alpha v\rangle \\
T_4 &= \langle r_\alpha A v - T_\alpha A v, u_\alpha - r_\alpha v\rangle \\
T_5 &= \langle T_\alpha A v - T_\alpha A r_\alpha v, u_\alpha - r_\alpha v\rangle.
\end{aligned}$$

Looking at these expressions individually and keeping in mind the boundedness of $\{u_\alpha\}$, we get from Lemma 23, Lemma 17 and Lemma 16, respectively, that as $\alpha \to 0$

$$\begin{aligned}
|T_1| &\leq (1 + \nu + (\kappa_2 + c)\alpha)\ |B_\alpha|\ \|u_\alpha\|\,\|u_\alpha - r_\alpha v\| \to 0, \\
|T_2 + T_4| &\leq\ \|(T_\alpha A - r_\alpha A)(\bar{u} - v)\|\,\|u_\alpha - r_\alpha v\| \\
&\leq n 2^{-n}\ |B_\alpha|\ \|\nabla_1 k\|_\infty\,\|\bar{u} - v\|\,\|u_\alpha - r_\alpha v\| \to 0, \\
|T_5| &\leq\ \|T_\alpha A\|_{\mathcal{L}(X)}\,\|(\mathrm{id} - r_\alpha) v\|\,\|u_\alpha - r_\alpha v\| \to 0.
\end{aligned}$$

To analyze the asymptotic behaviour of T_3 we note that

$$\begin{aligned}
|\langle f - Av, u_\alpha - v\rangle - \langle r_\alpha(f - Av), u_\alpha - r_\alpha v\rangle| &=\ |\langle (\mathrm{id} - r_\alpha)(f - Av), u_\alpha - v\rangle| \\
&\leq\ \|(\mathrm{id} - r_\alpha)(f - Av)\|\,\|u_\alpha - v\| \\
&\to 0 \quad \text{as} \quad \alpha \to 0.
\end{aligned}$$

Thus, the weak convergence, $u_\alpha \rightharpoonup u$, yields

$$\begin{aligned}
T_3 &= \langle f - Av, u_\alpha - v\rangle + \langle r_\alpha f - r_\alpha A v, u_\alpha - r_\alpha v\rangle - \langle f - Av, u_\alpha - v\rangle \\
&\to \langle f - Av, u - v\rangle \quad \text{as} \quad \alpha \to 0.
\end{aligned}$$

□

Proposition 30. *Let $\alpha_k \to 0$, then the sequence $u_k = u_{\alpha_k} \rightharpoonup \mathcal{M}$ as defined in* (3).

Proof. From Lemmas 28, 25 and 23 we see that

$$\|u_\alpha - \bar{u}\| \leq\ \|\big((T_\alpha A)_g + a_\alpha \mathrm{id}\big)^{-1}\|\,\|(T_\alpha A)_\ell - a_\alpha \mathrm{id}\|\,\|\bar{u}\| \leq C(\alpha)\,\|\bar{u}\|,$$

where $C(\alpha)$ is uniformly bounded. Therefore the family $\{u_k\}$ is bounded and we can extract a subsequence, denoted again by $\{u_k\}$, such that $u_k \rightharpoonup u \in X_\varepsilon$. To show that the weak limit u is indeed a solution of (1) we make use of the maximal monotonicity of A and it remains to show that

$$\langle f - Av, u - v\rangle \geq 0, \qquad \forall v \in X.$$

Writing $T_k = T_{\alpha_k}, r_k = r_{\alpha_k}$ we have for arbitrary $v \in X$ using Lemma 14 that

$$\begin{aligned} 0 &\le \langle (T_k A T_k^*)(u_k - r_k v), u_k - r_k v\rangle \\ &\le \langle (T_k A T_k^* - T_k A)(u_k - r_k v), u_k - r_k v\rangle + \langle T_k A(u_k - r_k v), u_k - r_k v\rangle \\ &\le \alpha_k \left|B_{\alpha_k}\right| n 2^{-n} \left\|\nabla_2 k\right\|_\infty \left\|u_k - r_k v\right\|^2 + \langle T_k A(u_k - r_k v), u_k - r_k v\rangle. \end{aligned}$$

Thus, it follows from Lemma 29 that

$$0 \le \lim_{k\to\infty} \langle (T_k A T_k^*)(u_k - r_k v), u_k - r_k v\rangle = \langle f - Av, u - v\rangle,$$

which completes the proof. The same reasoning can be applied to any subsequence of the original $\{u_k\}$ and thus the whole sequence weakly converges to $\mathcal{M}$.

□

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