PRELIMINARY REPORT ON THE RESEARCH CONDUCTED UNDER MARSHALL PLAN SCHOLARSHIP

JERNEJ ČINČ
(UNIVERSITÄT WIEN)

ABSTRACT. A long-standing open problem in Continuum Theory closely related with the fixed point problem reads as follows. Does there exist a planar continuum which admits a simple dense canal in every of its planar embeddings? In this document we propose a continuum which could answer the question in the affirmative.

1. ACKNOWLEDGEMENTS

I am very grateful to Professor Dr. Piotr Minc from Auburn University for all the given help, support and especially for a series of interesting mathematical discussions during my stay at Auburn University in Auburn, Alabama. I appreciate all the given help by the host university. Next, I express my gratitude to my advisor at Universiät Wien Professor Dr. Henk Bruin for the help with applying for the Marshall Plan Scholarship. I also thankfully acknowledge the financial support by the Austrian Marshall Plan Foundation. The given report is based on the preliminary version of a paper in progress written with Professor Dr. Piotr Minc.

2. INTRODUCTION

All the spaces considered in this document are going to be *metric spaces*. A *continuum* is a compact connected metric space. A continuum has a fixed point, if every continuous function of the continuum to itself (later in the document such function is called a *self-map*) has a fixed point. Continua with the property that every self-map has a fixed point are called to have the *fixed point property*. As an example, it follows directly from the intermediate-value theorem that the *unit interval* $[0,1] \subset \mathbb{R}$ has the fixed point property. A homeomorphism of a space onto a subspace of the plane is called a *planar embedding* of the space. A continuum that admits a planar embedding is called a *planar continuum*. We say that a planar continuum is *non-separating* if the complement of the continuum in the plane is connected. One of the oldest outstanding open questions in Continuum Theory is the following:

> (The Scottish Book, Problem 107, Sternbach, see [15]): Does every non-separating planar continuum have the fixed point property?

Brouwer's fixed point theorem states that a compact convex set has the fixed point property. The question quoted above has been one of the central topics of research in Continuum Theory ever since it was stated, since the positive answer on it would give a natural generalization of the Brouwer fixed point theorem in dimension two (for a survey on the fixed point property problem see [4, 11]). There have been a series of involved

Date: July 3, 2017.

examples of continua without the fixed point property with variety of additional topological properties given in the literature [3, 8, 10, 16, 17, 18, 19, 20, 24, 25]. However, no proof that the mentioned examples are planar was given and thus the fixed point property question remains unanswered.
A continuum is called *indecomposable*, if it cannot be written as a union of two of its proper subcontinua. A *topological disk* is a homeomorphic image of a closed unit planar ball. A technique to construct interesting indecomposable planar continua is to take a topological disk and dig in the disk an infinite canal with boundaries of the canal asymptotically approaching each other as the digged canal gets longer. If the closure of the boundaries of the canal within the disk is an indecomposable continuum, the canal is called a *simple dense canal* (in the literature also sometimes called a *Lakes-of-Wada channel*). A working definition of a simple dense canal is going to be given later. It was observed independently by Bell, Sieklucki and Iliadis [2, 12, 26] from 1967 until 1970 that an example of a continuum without the fixed point property (if such exists) needs to have an indecomposable continuum in its boundary. Furthermore, the results of the mentioned papers imply that an example of a non-separating planar continuum without the fixed point property (if such exists) would need to have a simple dense canal in every planar embedding of that continuum. Therefore it is natural to ask the following question which was posed in the paper [6] by Brechner and Mayer and restated in the Continuum Theory Problems paper [14] written by Lewis:

> (Problem 143 from [14], Brechner and Mayer): Does there exist a non-separating planar continuum such that every planar embedding of it has a simple dense canal?

To our knowledge no answer on the question by Brechner and Mayer has been given in the literature yet.

In this document we give a construction of a possible example of a continuum with a simple dense canal in every of its embeddings, which would provide a positive answer to the quoted question given by Brechner and Mayer. In the paragraphs to follow we describe the outline of the construction of the given example and we formalize this construction for the rest of the document.

An *arc* is a homeomorphic image of the closed unit interval. A *ray* is a homeomorphic image of $[0,1) \subset \mathbb{R}^2$. A ray contained in an planar embedding of an indecomposable continuum is said to have a *free side*, if for every subarc of the ray there exists an $\epsilon > 0$ so that exactly one side of the subarc in the ϵ-neighborhood of it contains no other points of the continuum. A *tree* is an acyclic graph.

Let X_n be continua and let $f_n : X_{n+1} \to X_n$ be continuous functions for every nonnegative integer n. The *inverse limit space* is defined by

$$\varprojlim \{X_n, f_n\}_{n=0}^{\infty} = \left\{(x_0, x_1, \dots) : x_n = f_{n+1}(x_{n+1}), x_n \in X_n \text{ for every nonnegative integer } n\right\}$$

and we call spaces X_n *factor spaces* and functions f_n *bonding maps*. For brevity we denote $X = \varprojlim \{X_n, f_n\}$. It is not difficult to see that the space X under given conditions is a continuum. We will use inverse limit construction as the main tool in the description of our example and we will refer to the example from now onwards by X. Our constructed continuum X is going to be a *tree-like continuum*, i.e. inverse limit space on trees as factor spaces. The continuum X is going to contain four distinct mutually disjoint rays R^v for $v \in \{0,1,2,3\}$ and each of the rays is going to be dense in X. All the rays R^v are going to have a free side in every planar embedding of X. The construction of the example will furthermore assure that the non-free sides of two pairs of rays R^0, R^1 and

R^2, R^3 respectively are going to face each other in every planar embedding of X. Therefore, our aim is to construct a simple dense canal between two free sides of rays R^v. A *simple triod* is a union of three arcs intersecting in a common endpoint and the three arcs are mutually disjoint otherwise. The continuum X is going to contain five distinct mutually disjoint simple triods; four of them are going to be attached to exactly one of the four rays R^v and one of the triods is going to be disjoint from all the rays R^v. The most important ingredient for the construction of a simple dense canal is going to be wrapping of all "long arcs" from X around the five mentioned simple triods. Intuitively, when some "long arc" from X will wrap on a simple triod we will be able to estimate the length of a subarc of this "long arc" that will stay close to the triod regardless of the planar embedding of X. Thus the subarcs of "long arcs" not staying close to any of the mentioned triods will not have sufficient length to prevent the existence of the simple dense canal in any planar embedding of X. Therefore, since we in such a way control all "long arcs" from X we will (hopefully) be able to build a simple dense canal between two free sides of rays R^v inductively on the lengths of the rays. At the end of the file we comment in Remark 11.1 what still needs to be done to complete the paper.

3. Preliminaries

In this section we define a language to describe bonding maps on the factor spaces for the inverse limit representation of our example.

Throughout this document let $\widehat{c'c''}$ denote an arc oriented from c' to c''. We denote by $\operatorname{int}\left(\widehat{c'c''}\right) = \widehat{c'c''} \setminus \{c', c''\}$.

Definition 3.1. Define a *walk on a tree D* as a finite sequence $W = (w_0, w_1, \dots, w_n)$ of points of D. When $W = (w)$ we omit brackets for brevity. Let $\ominus W$ denote the sequence W listed in the opposite order, i.e. $\ominus W = (w_n, w_{n-1}, \dots, w_0)$. For two walks $W' = \left(w'_0, w'_1, \dots, w'_n\right)$ and $W'' = \left(w''_0, w''_1, \dots, w''_{n'}\right)$, where n and n' are nonnegative integers, define $W' \oplus W'' = \left(w'_0, w'_1, \dots, w'_n, w''_0, w''_1, \dots, w''_{n'}\right)$. For walks $W_1, \dots, W_i$, define $\bigoplus_{j=1}^{i} W_j = W_1 \oplus \cdots \oplus W_i$. If $i = 0$ we understand that $\bigoplus_{j=1}^{i} W_j = \emptyset$. For any positive integer k, let W^k denote $W \oplus W \oplus \cdots \oplus W$ (the concatenation of k walks W). The walk W^* is the *abbreviation* of the walk W if all identical consecutive points from W are replaced by one such point.

Definition 3.2. Suppose W is a walk on a tree D. Let $W^* = (w_0, w_1, \dots, w_n)$ and $\widehat{c'c''}$ be an oriented arc. Denote by $c_0 = c', c_1, \dots, c_{n-1}, c_n = c''$ strictly increasing sequence of points from $\widehat{c'c''}$. Define the map $\alpha\langle W, \widehat{c'c''}, D\rangle : \widehat{c'c''} \to D$ by setting $\alpha\langle W, \widehat{c'c''}, D\rangle(c_j) = w_j$ and letting $\alpha\langle W, \widehat{c'c''}, D\rangle|_{\widehat{c_{j-1}c_j}}$ be an arbitrary homeomorphism onto $\widehat{w_{j-1}w_j}$ such that $\alpha\langle W, \widehat{c'c''}, D\rangle(c_{j-1}) = w_{j-1}$ and $\alpha\langle W, \widehat{c'c''}, D\rangle(c_j) = w_j$ for each $j \in \{1, \dots, n\}$.

4. Construction of factor spaces and bonding maps

In this section we first give geometric description of the factor spaces used in the construction of inverse limit and then in the following subsections define and explain the action of bonding maps on the factor spaces.

Suppose that T is a simple triod with endpoints t_0, t_1 and t_2. Let Y be another simple triod disjoint from T. Denote the endpoints of Y by y_0, y_1 and y_2. Let $\widehat{y_0a_0}$ be an arc intersecting $T \cup Y$ only at y_0. Let $\widehat{t_0a}$ be an arc containing a_0 in its interior and intersecting $Y \cup \widehat{y_0a_0} \cup T$ only at t_0 and a_0. Denote $Y \cup \widehat{y_0a_0} \cup \widehat{t_0a} \cup T$ by A_0.

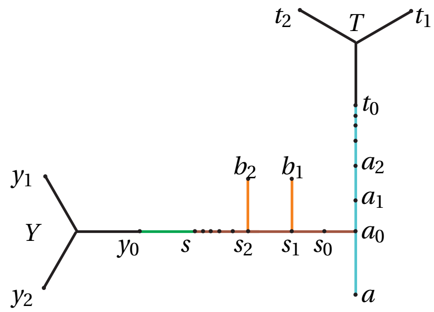


FIGURE 1. A_2

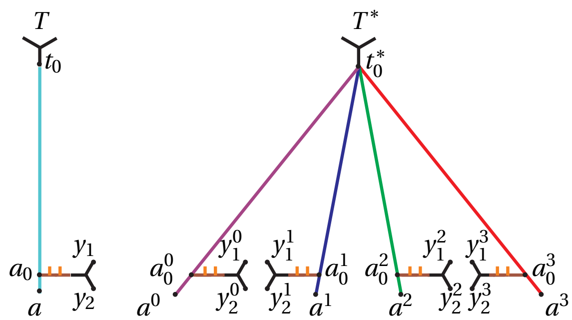


FIGURE 2. A_2 and $\widetilde{A}_2$

Let s be a point in the interior of $\widehat{a_0 y_0}$ and let $s_0, s_1, s_2, \dots$ be a strictly increasing sequence of points in the interior of $\widehat{a_0 y_0}$ so that $\lim_{i\to\infty} s_i = s$. Suppose $\widehat{s_1 b_1}, \widehat{s_2 b_2}, \widehat{s_3 b_3}, \dots$ is a sequence of mutually disjoint arcs such that $A_0 \cap \widehat{s_i b_i} = \{s_i\}$ for each positive integer i. Set $A_i = A_0 \cup \bigcup_{j=1}^{i} \widehat{s_j b_j}$.

Observation 4.1. $A_i \subset A_{i+1}$ for each nonnegative integer i.

Let $a_1, a_2, \dots$ be a strictly increasing sequence of points in the interior of $\widehat{a_0 t_0}$. Additionally, we assume that $\lim_{n\to\infty} a_n = t_0$. For each nonnegative integer n, let m_n be a point in the interior of $\widehat{a_n a_{n+1}}$. Set $\beta_n = \alpha \langle (a_n, t_0), \widehat{a_n m_n}, A_0\rangle$. Also, let G_n denote the set $\widehat{a a_n} \cup \widehat{a_0 s_0}$.

In the product $A_i \times \{0,1,2,3\}$ consider the relation $\sim$ defined by $(x, \mu) \sim (z, \nu)$ for $x, z \in A_i$ and $\mu, \nu \in \{0,1,2,3\}$ if and only if $x = z$ and either $\mu = \nu$, or $x = z \in T$. Let $\widetilde{A}_i$ denote the quotient space $A_i \times \{0,1,2,3\} / \sim$ and let $q_i : A_i \times \{0,1,2,3\} \to \widetilde{A}_i$ be the quotient map. We use the following notation for brevity. If $z \in A_i$ and $\mu \in \{0,1,2,3\}$ we denote $q_i\left((z,\mu)\right)$ by z^μ. If $Z \subset A_i$ we denote $q_i\left(Z \times \{\mu\}\right)$ by Z^μ. If $t \in T$ we denote $q_i\left((t,\mu)\right)$ by t^*. In the same convention, we use T^* for $q_i\left(T \times \{\mu\}\right)$.

Observation 4.2. $\widetilde{A}_i \subset \widetilde{A}_{i+1}$ for each nonnegative integer i.

We define an involution $\tau : \{0,1,2,3\} \to \{0,1,2,3\}$ by $\tau(0) = 1$, $\tau(1) = 0$, $\tau(2) = 3$ and $\tau(3) = 2$.

Let us comment on the important parts of the factor spaces $\widetilde{A}_i$ and establish a language so that we can address them. The space $\widetilde{A}_i$ consists of four trees A_i^ν called ν-*legs* for $\nu \in \{0,1,2,3\}$, where points $t \in T \subset A_i^\nu$ are identified, see Figure 2. For every $\nu \in \{0,1,2,3\}$ a ν-leg consists of an arc $\widehat{a^\nu t_0^*}$ and a tree $\widehat{a_0^\nu y_0^\nu} \bigcup_{j=0}^{i} \widehat{s_i^\nu b_i^\nu} \cup Y^\nu$. A subarc $\widehat{a^\nu a_{n+1}^\nu}$ of $\widehat{a^\nu t_0^*}$ is called the ν*-precursor of a ray*. The arc $\widehat{s_i^\nu b_i^\nu}$ is called the i*-sticker of the* ν*-leg*.

In the next two subsections we will separately define the maps $\varphi_{n,k} : \widetilde{A}_i \to \widetilde{A}_i$ for n odd and $i = \frac{n+1}{2}$ and $\varphi_{n,k} : \widetilde{A}_{i+1} \to \widetilde{A}_i$ for n even and $i = \frac{n}{2}$ and any positive integer k. From now onwards k will always denote the number of wrappings of stretched ν-precursors of rays around a triod from $\widetilde{A}_i$ and is going to be refereed to as *wrapping number*. The maps $\varphi_{n,k}$ are going to be used as bonding maps in the definition of our inverse limit space X. We will prove later that for a careful inductive choice of wrapping numbers k, any planar embedding of X has a simple dense canal. Before we give a formal definition of maps $\varphi_{n,k}$ we first intuitively describe what we require from the bonding maps.

With $\varphi_{n,k}$ for odd n we will assure that X is having four distinct mutually disjoint rays R^ν which will correspond to the union of extensions of ν-precursors of rays for every $\nu \in \{0,1,2,3\}$ (we will define rays R^ν precisely later in the document). The rays R^ν are going to be dense in X, which is going to be achieved by stretching the precursors of rays along every ν-leg for $\nu \in \{0,1,2,3\}$. Furthermore, we want that the maps $\varphi_{n,k}$ for both even and odd n assure the existence of a free side for every of four rays R^ν in every planar embedding of X. We will achieve that with fixing the triods $\widehat{a^\nu a_1^\nu} \cup \widehat{a_0^\nu s_0^\nu}$ for every $\nu \in \{0,1,2,3\}$ with every map $\varphi_{n,k}$ and extending ν-precursors of rays along the side of the arc $\widehat{a^\nu a_1^\nu}$ which contains a subarc of $\widehat{a_0^\nu s_0^\nu}$. The map $\varphi_{n,k}$ for even n is going to extend the i-sticker of the μ-leg on the part of the $\tau(\mu)$-leg and wrap it k times around triod $Y^{\tau(\mu)}$ and vice versa for the i-sticker of the $\tau(\mu)$-leg for $\mu \in \{0,2\}$. Extended stickers are going to be introduced to assure that the non-free sides of R^μ and $R^{\tau(\mu)}$ face each other in every planar embedding of X for $\mu \in \{0,2\}$. Intuitively, the extended stickers will tie together pairs of rays R^μ and $R^{\tau(\mu)}$ for $\mu \in \{0,2\}$. Thus we will be able to start the construction of a simple dense canal of a planar embedding of X between free sides of two rays. Moreover, $\varphi_{n,k}$ for n odd will introduce wrapping of extensions of ν-precursors of rays k times around triods Y^ν for every $\nu \in \{0,1,2,3\}$ and the map $\varphi_{n,k}$ for n even wrapping of extensions of ν-precursors of rays k times around triod T^*. A careful inductive choice of wrapping numbers around all the triods is going to be of main importance in the construction of simple dense canal in an arbitrary planar embedding of X, as explained in the introduction.

4.1. **Construction of the map $\varphi_{n,k}$ for odd n.** Throughout this subsection k is a positive integer, $n \geq 0$ is an odd integer, $i = \frac{n+1}{2}$, $\mu = \mathrm{mod}\left(\frac{n-1}{2}, 4\right)$ (i.e. $\frac{n-1}{2} = 4l + \mu$ for the nonnegative integer l so that $\mu \in \{0,1,2,3\}$), and ν is an arbitrary element of $\{0,1,2,3\}$.

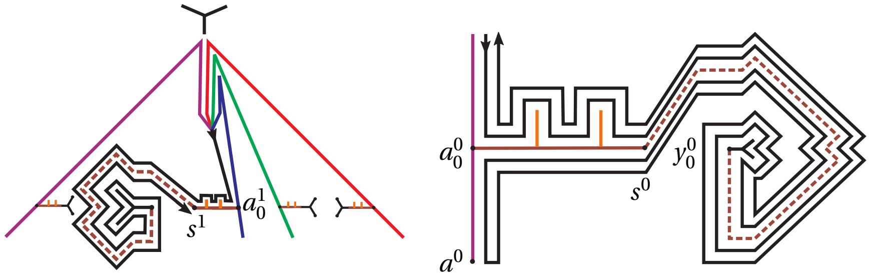
Let $P = \left(t_0^*, a_0^\mu\right) \oplus \bigoplus_{j=1}^{i} \left(s_j^\mu, b_j^\mu, s_j^\mu\right) \oplus \left(y_1^\mu, y_2^\mu, y_0^\mu\right)^k \oplus \left(y_2^\mu, y_1^\mu, y_0^\mu\right)^{k-1} \oplus a^\mu$. Let γ_o denote $\alpha \langle P \ominus P, \widehat{m_n a_{n+1}}, \widetilde{A}_i \rangle$.

We define a mapping $\varphi_{n,k} : \widetilde{A}_i \to \widetilde{A}_i$ in the following way.

(O-1) Let $\varphi_{n,k}(x^\nu) = x^\nu$ if $x \in Y \cup \widehat{sa_0} \cup \bigcup_{j=1}^{i} \widehat{s_j b_j} \cup \widehat{aa_n} \cup T$.

(O-2) Let $\varphi_{n,k}(x^\nu) = x^\nu$ if $\nu \neq \mu$ and $x \in \widehat{sy_0}$.

(O-3) Let $\varphi_{n,k}\big|_{\widehat{s^\mu y_0^\mu}} = \alpha \langle W, \widehat{s^\mu y_0^\mu}, \widetilde{A}_i \rangle$, where $W = s^\mu \oplus \left(y_1^\mu, y_2^\mu, y_0^\mu\right)^{k-1}$.



(i) (ii)

FIGURE 3. $\varphi_{3,2}\left(\widetilde{A}_2\right)$ with a 1-gap. An extended ν-precursor of a ray is approaching the μ-leg ($\mu = 1$ in case (i) and $\mu = 0$ in case (ii)).

(O-4) Let $\varphi_{n,k}(x^\nu) = q\left(\beta_n(x), \nu\right)$ if $x \in \widehat{a_n m_n}$.
(O-5) Let $\varphi_{n,k}(x^\nu) = \gamma_0(x)$ if $x \in \widehat{m_n a_{n+1}}$.
(O-6) Let $\varphi_{n,k}(x^\nu) = t_0^*$ for each $x \in \widehat{a_{n+1} t_0}$.

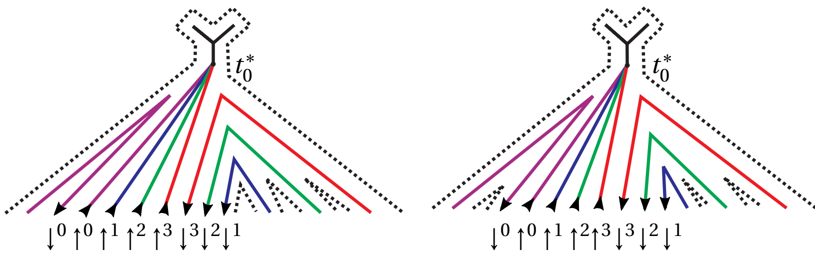
Observation 4.3. $\varphi_{n,k}$ is continuous for odd n.

Let $\Gamma_{\varphi_{n,k}}$ denote the *graph* of $\varphi_{n,k} : \widetilde{A}_i \to \widetilde{A}_i$. In the following paragraphs we explain in detail the construction of the map $\varphi_{n,k}$ for odd n and any positive integer k. We argue that the graph $\Gamma_{\varphi_{n,k}}$ can be drawn in the plane arbitrarily close to $\widetilde{A}_i$ for every odd n and any positive integer k.

First, note that the points $x^\nu \in \widetilde{A}_i \setminus \left(\bigcup_{\nu=0}^{3} \widehat{a_n^\nu t_0^*} \cup \widehat{s^\mu y_0^\mu}\right)$ can be drawn arbitrary close to $\varphi_{n,k}(x^\nu) = x^\nu$ by (O-1), (O-2) for every positive integer k and odd n. By (O-3) we draw $\Gamma_{\varphi_{n,k}}|_{\widehat{s^\mu y_0^\mu}}$ to wrap $k-1$ times around Y^μ and thus creating a $(k-1)$-*gap* between the points s^μ and y_0^μ around the triod Y^μ (see the dashed line on the Figure 3). The $(k-1)$-gap is formed to allow the stretched all ν-precursors of rays to wrap around Y^μ. Arcs which enter this $(k-1)$-gap can be drawn to wrap at most $k-1$ times around Y^μ for $\mu = 0, 2$ in the clockwise direction and at most $k-1$ times around Y^μ for $\mu = 1, 3$ in counterclockwise direction. The ν-precursors of rays are by (O-4) first extended from $\widehat{a^\nu m_n^\nu}$ to the entire arc $\widehat{a^\nu t_0^\nu}$ for every $\nu \in \{0,1,2,3\}$. After observations made this paragraph the graph $\Gamma_{\varphi_{n,k}}$ after applying (O-1)-(O-4) looks as on Figure 3, case (i).

The arcs $\widehat{m_n^\nu a_{n+1}^\nu}$ are by (O-5) stretched by $\varphi_{n,k}$ along the whole leg A_i^μ, starting from t_0^*, continuing to a_0^μ and passing around the j-stickers of μ-leg for every $j = 1, \dots, i$, see Figure 3. Still by (0-5) we draw $\Gamma_{\varphi_{n,k}}|_{\widehat{m^\nu a_{n+1}^\nu}}$ to enter the $(k-1)$-gap and wrap inside it $k-1$ times around Y^μ, unwrap inside it $k-1$ times and exit the $(k-1)$-gap under the arc $\Gamma_{\varphi_{n,k}}|_{\widehat{s^\mu y_0^\mu}}$. Note that in such a way a subarc of $\Gamma_{\varphi_{n,k}}|_{\widehat{m_n^\nu a_{n+1}^\nu}}$ indeed wraps around Y^μ exactly k times, since it wraps once around Y^μ before entering the $(k-1)$-gap, see Figure 3. After unwrapping, a subarc of $\Gamma_{\varphi_{n,k}}|_{\widehat{m_n^\nu a_{n+1}^\nu}}$ stretches under the arc $\widehat{a_0^\mu s^\mu}$ to the point a^μ and then does all of the movement described in this paragraph in the reverse order and finally stretches to the point t_0^*, see Figure 3, case (ii).

Since $\varphi_{n,k}$ stretches ν-precursors of rays for every $\nu \in \{0,1,2,3\}$ to a μ-leg in the way described above, we need to show that the arcs $\Gamma_{\varphi_{n,k}}|_{\widehat{m_n^\nu a_{n+1}^\nu}}$ can indeed be drawn simultaneously in the plane so that $\varphi_{n,k}\left(a_{n+1}^\nu\right) = t_0^*$ for every $\nu \in \{0,1,2,3\}$ as required by (O-6).



(i) (ii)

FIGURE 4. Part of the graph $\Gamma_{\varphi_{n,k}}$ for $\mu = 0$ in case (i) and for $\mu = 1$ in case (ii).

Observation 4.4. By the (O-5) there exists a unique point $u_n^\nu \in \operatorname{int}\left(\widehat{m_n^\nu a_{n+1}^\nu}\right)$ such that $\varphi_{n,k}\left(u_n^\nu\right) = a^\mu$ for every $\nu \in \{0,1,2,3\}$.

We impose an ordering on the planar arcs $\Gamma_{\varphi_{n,k}}|_{\widehat{m_n^\nu a_{n+1}^\nu}}$. Denote by $\downharpoonleft^\nu = \Gamma_{\varphi_{n,k}}|_{\widehat{m_n^\nu u_n^\nu}}$ and by $\upharpoonleft^\nu = \Gamma_{\varphi_{n,k}}|_{\widehat{u_n^\nu a_{n+1}^\nu}}$ for every $\nu \in \{0,1,2,3\}$ and let $\mathscr{P}^\mu = \left\{\updownarrow^\nu : \nu \in \{0,1,2,3\} \text{ and } \updownarrow \in \{\upharpoonleft, \downharpoonleft\}\right\}$ for every $\mu \in \{0,1,2,3\}$. We write $\updownarrow^\nu \prec \updownarrow^\lambda$, if arc $\updownarrow^\nu$ is drawn in the plane closer to μ-leg than to $\updownarrow^\lambda$ among two different arcs $\updownarrow^\nu, \updownarrow^\lambda \in \mathscr{P}^\mu$. We study different cases depending on the choice of $\mu \in \{0,1,2,3\}$.

Let $\mu = 0$. Since the graph $\Gamma_{\varphi_{n,k}}$ should be drawn in the plane, it follows that $\downharpoonleft^0$ needs to be drawn the closest to the 0-leg and that $\downharpoonleft^3 \prec \downharpoonleft^2 \prec \downharpoonleft^1$ and no other arc from $\mathscr{P}^0$ is between arcs $\downharpoonleft^\nu$ for $\nu \in \{1,2,3\}$ since $\varphi_{n,k}\left(a_{n+1}^\nu\right) = t_0^*$ by (O-6), see Figure 4, case (i). Furthermore the arc $\downharpoonleft^1$ obviously needs to be drawn the furthest away from A_i^0 among all the elements from $\mathscr{P}^0$. Thus we only need to determine the ordering among the arcs $\downharpoonleft^0 \prec \upharpoonleft^0, \upharpoonleft^1, \upharpoonleft^2, \upharpoonleft^3 \prec \downharpoonleft^3$.

Observation 4.5. Let l be a positive integer. Suppose that $(0,z_0), (0,z_1), \ldots, (0,z_l) \in [0,1] \times [0,1] \subset \mathbb{R}^2$ is a sequence of points so that $0 = z_0 < z_1 < \cdots < z_l = 1$ and $\left(1, z_0'\right), \ldots, \left(1, z_l'\right) \in [0,1] \times [0,1]$ and let $Z_0, Z_1, \ldots, Z_l$ be arcs in $[0,1] \times [0,1]$ such that the endpoints of Z_j are $\left(0, z_j\right)$ and $\left(1, z_j'\right)$ for every $j \in \{0,1,\ldots,l\}$. If arcs Z_j are mutually disjoint, then $z_0' < z_1' < \cdots < z_l'$.

Applying Observation 4.5 for Z_ν being proper subarcs of $\downharpoonleft^\nu \cup \upharpoonleft^\nu$ for every $\nu \in \{0,1,2,3\}$ with a planar homeomorphism on $[0,1] \times [0,1]$ we obtain that $\downharpoonleft^0 \prec \upharpoonleft^0 \prec \upharpoonleft^1 \prec \upharpoonleft^2 \prec \upharpoonleft^3 \prec \downharpoonleft^3$, which completely determines the ordering on arcs from $\mathscr{P}^0$.

Let $\mu = 1$. Since the graph $\Gamma_{\varphi_{n,k}}$ should be drawn in the plane, it follows that $\downharpoonleft^1$ needs to be drawn the closest to the 1-leg and furthermore we have the order $\downharpoonleft^3 \succ \downharpoonleft^2 \succ \downharpoonleft^1$ on these arcs. Moreover, no other arc from $\mathscr{P}^1$ is between arcs $\downharpoonleft^3$, $\downharpoonleft^2$ and $\downharpoonleft^1$, since $\varphi_{n,k}(a_{n+1}) = t_0^*$ by (O-6), see Figure 4, case (ii). Furthermore, again by (O-6) the arc $\downharpoonleft^0$ needs to be drawn the furthest away from the 1-leg among all the elements of $\mathscr{P}^1$. Thus we only need to determine the ordering on $\downharpoonleft^0 \succ \upharpoonleft^0, \upharpoonleft^1, \upharpoonleft^2, \upharpoonleft^3 \succ \downharpoonleft^3$. Applying Observation 4.5 again for Z_ν being subarcs of $\downharpoonleft^\nu \cup \upharpoonleft^\nu$ for every $\nu \in \{0,1,2,3\}$ with a planar homeomorphism on $[0,1] \times [0,1]$ we obtain that $\downharpoonleft^0 \prec \upharpoonleft^0 \prec \upharpoonleft^1 \prec \upharpoonleft^2 \prec \upharpoonleft^3 \prec \downharpoonleft^3$, see Figure 4, case (ii).

If we interchange in $\widetilde{A}_i$ the 0-leg with the 3-leg and the 1-leg with the 2-leg (i.e. reflect $\widetilde{A}_i$ over the vertical line of symmetry of $\widetilde{A}_i$) the graph $\Gamma_{\varphi_{n,k}}$ for n odd and for either $\mu = 3$ or $\mu = 2$ respectively can be drawn analogously as discussed above for either $\mu = 0$ or

$\mu = 1$ respectively. We have studied all the elements of the definition of map $\varphi_{n,k}$ for odd n and positive integer k. We conclude that the graph $\Gamma_{\varphi_{n,k}}$ can be drawn arbitrarily close in the plane to $\widetilde{A}_i$.

We continue with some observations that are going to be important later in the document.

Observation 4.6. $\varphi_{n,k}$ restricted to G_n^ν is the identity on G_n^ν. Additionally, G_n^ν is a component of ${\varphi_{n,k}}^{-1}\left(G_n^\nu\right)$.

Observation 4.7. Let L be an arc contained either in $\operatorname{int}\left(\widehat{a_0^\nu a_n^\nu}\right)$ or in $\operatorname{int}\left(\widehat{s_j^\nu b_j^\nu}\right)$ for some $j = 1,\dots,i$ and let K be a component of ${\varphi_{n,k}}^{-1}(L)$. Then K is either L or an arc contained in $\operatorname{int}\left(\widehat{m_n^\lambda a_{n+1}^\lambda}\right)$ for some $\lambda \in \{0,1,2,3\}$. In both cases $\varphi_{n,k}$ restricted to K is a homeomorphism of K onto L. (Notice that the case $K \neq L$ may occur only when $\nu = \mu$.)

Observation 4.8. Let $c^\nu \in G_n^\nu \setminus \left\{s_0^\nu\right\}$. Then the conclusion of the above observation is also true if $L = \widehat{c^\nu s_0^\nu}$ and K is a component of ${\varphi_{n,k}}^{-1}(L)$ such that $c^\nu \in \varphi_{n,k}(K)$.

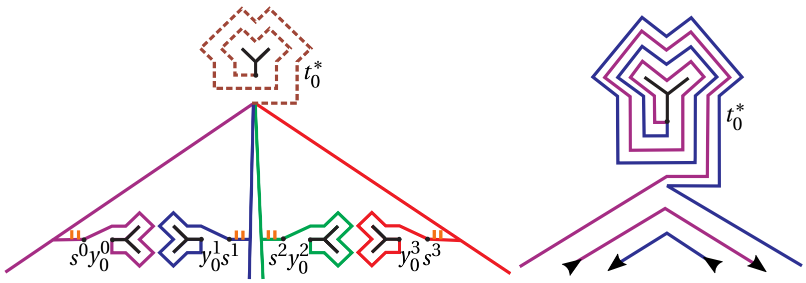
4.2. **Construction of the map $\varphi_{n,k}$ for even n.** Throughout this subsection ν is an arbitrary element of $\{0,1,2,3\}$, k is a positive integer, n is an even nonnegative integer and $i = \frac{n}{2}$. In the case of even n, $\varphi_{n,k} : \widetilde{A}_{i+1} \to \widetilde{A}_i$. Before we define this mapping, we need to introduce the following notation:

- Set, $\gamma_{\mathrm{e}} = \alpha\langle t_0^* \oplus \left(t_1^*, t_2^*, t_0^*\right)^k, \widehat{m_n a_{n+1}}, \widetilde{A}_i\rangle$.
- Let $\underline{b}_{i+1}^\nu$ be a point in the interior of $\widehat{s_{i+1}^\nu b_{i+1}^\nu}$.
- If $\nu \in \{1,3\}$, let $\underline{s}_{i+1}^\nu$ be a point in the interior of $\widehat{s_{i+1}^\nu \underline{b}_{i+1}^\nu}$.

In the remaining part of this subsection μ stands for an arbitrary element of $\{0,2\}$. (Thus, $\tau(\mu) \in \{1,3\}$.) We define a mapping $\varphi_{n,k} : \widetilde{A}_{i+1} \to \widetilde{A}_i$ in the following way.

(E-1) Let $\varphi_{n,k}(x^\nu) = x^\nu$ for each $x \in A_{i+1} \setminus \left(\widehat{sy_0} \cup \widehat{a_n t_0} \cup \widehat{s_{i+1} b_{i+1}}\right)$. (Since $x \in A_{i+1} \setminus \widehat{s_{i+1} b_{i+1}}$, it follows $x \in A_i$ and $\varphi_{n,k}(x^\nu) = x^\nu \in \widetilde{A}_i$.)
(E-2) Let $\varphi_{n,k}(x^\nu) = q\left(\beta_n(x), \nu\right)$ if $x \in \widehat{a_n m_n}$.
(E-3) Let $\varphi_{n,k}(x^\nu) = \gamma_{\mathrm{e}}(x)$ if $x \in \widehat{m_n a_{n+1}}$.
(E-4) Let $\varphi_{n,k}(x^\nu) = t_0^*$ for each $x \in \widehat{a_{n+1} t_0}$.
(E-5) Let $\varphi_{n,k}\big|_{\widehat{s^\nu y_0^\nu}} = \alpha\langle s^\nu \oplus \left(y_1^\nu, y_2^\nu, y_0^\nu\right)^{k-1}, \widehat{s^\nu y_0^\nu}, \widetilde{A}_i\rangle$.
(E-6) Let $\varphi_{n,k}\big|_{\widehat{s_{i+1}^\mu \underline{b}_{i+1}^\mu}} = \alpha\langle Q_1, \widehat{s_{i+1}^\mu \underline{b}_{i+1}^\mu}, \widetilde{A}_i\rangle$, where $Q_1 = s_{i+1}^\mu \oplus \bigoplus_{j=i}^1 \left(s_{j+1}^\mu, s_j^\mu, b_j^\mu, s_j^\mu\right) \oplus \left(a_0^\mu, t_0^*, a_0^{\tau(\mu)}, s_1^{\tau(\mu)}\right) \oplus \bigoplus_{j=1}^i \left(s_j^{\tau(\mu)}, b_j^{\tau(\mu)}, s_j^{\tau(\mu)}, s_{j+1}^{\tau(\mu)}\right) \oplus y_0^{\tau(\mu)}$.
(E-7) Let $\varphi_{n,k}\big|_{\widehat{\underline{b}_{i+1}^\mu b_{i+1}^\mu}} = \alpha\langle Q_2, \widehat{\underline{b}_{i+1}^\mu b_{i+1}^\mu}, \widetilde{A}_i\rangle$, where $Q_2 = y_0^{\tau(\mu)} \oplus \left(y_1^{\tau(\mu)}, y_2^{\tau(\mu)}, y_0^{\tau(\mu)}\right)^k$.
(E-8) Let $\varphi_{n,k}\big|_{\widehat{s_{i+1}^{\tau(\mu)} \underline{s}_{i+1}^{\tau(\mu)}}} = \alpha\langle V_1, \widehat{s_{i+1}^{\tau(\mu)} \underline{s}_{i+1}^{\tau(\mu)}}, \widetilde{A}_i\rangle$, where $V_1 = s_{i+1}^{\tau(\mu)} \oplus \left(y_1^{\tau(\mu)}, y_2^{\tau(\mu)}, y_0^{\tau(\mu)}\right)^k \oplus \left(y_2^{\tau(\mu)}, y_1^{\tau(\mu)}, y_0^{\tau(\mu)}\right)^k$.
(E-9) Let $\varphi_{n,k}\big|_{\widehat{\underline{s}_{i+1}^{\tau(\mu)} \underline{b}_{i+1}^{\tau(\mu)}}} = \alpha\langle V_2, \widehat{\underline{s}_{i+1}^{\tau(\mu)} \underline{b}_{i+1}^{\tau(\mu)}}, \widetilde{A}_i\rangle$, where $V_2 = y_0^{\tau(\mu)} \oplus \bigoplus_{j=i}^1 \left(s_{j+1}^{\tau(\mu)}, s_j^{\tau(\mu)}, b_j^{\tau(\mu)}, s_j^{\tau(\mu)}\right) \oplus \left(a_0^{\tau(\mu)}, t_0^*, a_0^\mu, s_1^\mu\right) \oplus \bigoplus_{j=1}^i \left(s_j^\mu, b_j^\mu, s_j^\mu, s_{j+1}^\mu\right) \oplus y_0^\mu$.
(E-10) Let $\varphi_{n,k}\big|_{\widehat{\underline{b}_{i+1}^{\tau(\mu)} b_{i+1}^{\tau(\mu)}}} = \alpha\langle V_3, \widehat{\underline{b}_{i+1}^{\tau(\mu)} b_{i+1}^{\tau(\mu)}}, \widetilde{A}_i\rangle$, where $V_3 = y_0^\mu \oplus \left(y_1^\mu, y_2^\mu, y_0^\mu\right)^k$.

Observation 4.9. $\varphi_{n,k}$ is continuous also for even n.



(i) (ii)

FIGURE 5. Case (i): graph $\Gamma_{\varphi_{4,2}}$ (and thus $i = 3$) for (E-1)-(E-5). Case (ii): graph $\Gamma_{\varphi_{n,k}}$ for $\mu = 0$ (dashed lines) and $\tau(\mu) = 0$ around the triod T^*.

In this subsection $\Gamma_{\varphi_{n,k}}$ refers to the graph of $\varphi_{n,k} : \widetilde{A}_{i+1} \to \widetilde{A}_i$. In the paragraphs to follow we explain the construction of the map $\varphi_{n,k}$ for even n and any positive integer k. We argue that the graph $\Gamma_{\varphi_{n,k}}$ can be drawn in the plane arbitrarily close to $\widetilde{A}_i$ independently of k and n.

By (E-1), if $x \in A_{i+1} \setminus \left(\widehat{sy_0} \cup \widehat{a_n t_0} \cup \widehat{s_{i+1} b_{i+1}} \right)$, then $\varphi_{n,k}(x^\nu) = x^\nu \in \widetilde{A}_i$ and thus graph $\Gamma_{\varphi_{n,k}}$ can be drawn arbitrarily close to $\widetilde{A}_i$ in this case. By (E-2) arcs $\widehat{a_n^\nu m_n^\nu}$ are with $\varphi_{n,k}$ stretched homeomorphically to $\widehat{a_n^\nu t^*}$ and thus the wrapping of arcs $\Gamma_{\varphi_{n,k}}|_{\widehat{m_n^\nu a_{n+1}^\nu}}$ around T^* by (E-3) can start close to the point t_0^*. By (E-3) we draw arcs $\Gamma_{\varphi_{n,k}}|_{\widehat{m_n^\nu a_{n+1}^\nu}}$ wrapping counterclockwise k times around the triod T^* (see the dashed line on Figure 5, case (i) which represent simultaneous parallel wrapping of all four arcs $\Gamma_{\varphi_{n,k}}|_{\widehat{m_n^\nu a_{n+1}^\nu}}$ around T^*). Furthermore, by (E-5) we draw arcs $\Gamma_{\varphi_{n,k}}|_{\widehat{s^\nu y_0^\nu}}$ wrapping around Y^ν which creates a $(k-1)$-gap in the clockwise direction around Y^ν for $\nu \in \{0,2\}$ and in the counterclockwise direction around Y^ν for $\nu \in \{1,3\}$ for any positive integer k. By the observations made in this paragraph, the graph $\Gamma_{\varphi_{n,k}}$ after applying (E-1)-(E-5) looks as on Figure 5, case (i).

Observation 4.10. If $x \in A_{i+1} \setminus \widehat{s_{i+1} b_{i+1}}$, it holds that $x^\nu \in \widetilde{A}_i$.

By 4.10, what remains to be discussed is the action of $\varphi_{n,k}$ on the $(i+1)$-stickers of ν-legs $\widehat{s_{i+1}^\nu b_{i+1}^\nu}$ for $\nu \in \{0,1,2,3\}$. Note that for every $x^\nu \in \widehat{s_{i+1}^\nu b_{i+1}^\nu}$ point $\varphi_{n,k}(x^\nu) \in A_i^\nu$ or $\varphi_{n,k}(x^\nu) \in A_i^{\tau(\nu)}$. Therefore, for discussing (E-6)-(E-10) it is sufficient to restrict on the case $\mu = 0$ and $\tau(\mu) = 1$ since the case for $\mu = 2$ and $\tau(\mu) = 3$ follows analogously. We refer to $\Gamma_{\varphi_{n,k}}|_{\widehat{s_{i+1}^0 b_{i+1}^0}}$ by a *simple bridge* and to $\Gamma_{\varphi_{n,k}}|_{\widehat{s_{i+1}^1 b_{i+1}^1}}$ by a *complicated bridge*. The simple bridge starts in the point s_{i+1}^0, stretches close to a_0^0, and passes around the j-stickers of 0-leg, for $j = i, \ldots, 1$ consecutively (see Figure 6, case (i)). Then the bridge stretches to t_0^* (see Figure 5, case (ii)) and stretches down the 1-leg close to the point a_0^1 and then passes around j-stickers of the 0-leg, for $j = 1, \ldots, i$ consecutively, see Figure 6, case (ii). We have already observed that there exists a $(k-1)$-gap around Y^1. Up to now we were describing the definition of the simple bridge by (E-6). Then, by (E-7) the simple bridge starts to wrap in the counterclockwise direction around Y^1 above the arc $\Gamma_{\varphi_{n,k}}|_{\widehat{s^1 y_0^1}}$ and is drawn to wrap k times around Y^1 and ends approaching the point y_0^1, which is possible since there is a $(k-1)$-gap around Y^1, see Figure 6, case (ii).

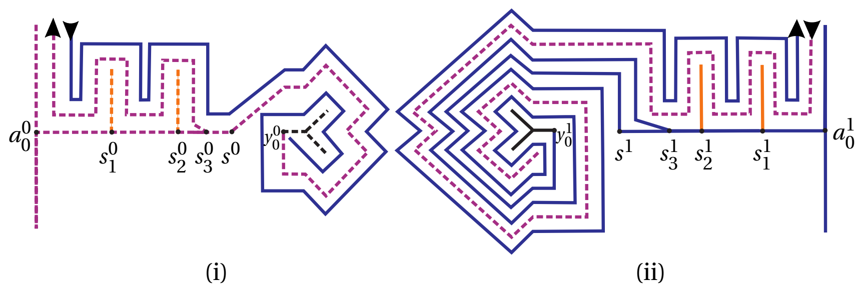


FIGURE 6. The graph $\Gamma_{\varphi_{4,2}}$ around triods Y^0 and Y^1. The dashed lines are parts of $\Gamma_{\varphi_{4,2}}|_{A_2^0}$.

Note that once the simple bridge is drawn in the plane as we described, we can not symmetrically reflect the construction of the simple bridge on the bridge starting from the point s_{i+1}^1, since such bridge would enter the dead end bounded by the simple bridge and the arc $\Gamma_{\varphi_{n,k}}|_{\widehat{s_{i+1}^0 s_{i+1}^1}}$, where $\widehat{s_{i+1}^0 s_{i+1}^1} \subset \widetilde{A}_{i+1}$. Thus, such bridge could not be drawn in the plane to wrap around Y^0 as the definition (E-10) requires.
Now we explain the construction of the complicated bridge. By (E-8) the complicated bridge starting from s_{i+1}^1 first wraps in the counterclockwise direction around Y^1 above the arc $\Gamma_{\varphi_{n,k}}|_{\widehat{s^1 y_0^1}}$. Since there is a $(k-1)$-gap around Y^1 the complicated bridge can be drawn to wrap all together k times around Y^1 and ends wrapping close to the point y_0^1 (as required by (E-8), see Figure 6, case (ii)). Then (still by (E-8)), the complicated bridge unwraps around Y^1 in the clockwise direction k-times and exits the $(k-1)$-gap on the opposite side of the simple bridge as it started, again see Figure 6, case (ii). By (E-9) the bridge passes around j-stickers of 1-leg, for $j = i, \dots, 1$ consecutively, approaches first a_0^1, then t_0^* and stretches down the 0-leg where it passes around all the j-stickers of 0-leg, for $j = 1, \dots, i$ consecutively. Since there is a $(k-1)$-gap around Y^0 the complicated bridge can be drawn to wrap k times around Y^0 as given in the definition (E-10) and ends approaching the point y_0^0, see Figure 6, case (i).

We have commented all the elements of the map $\varphi_{n,k}$ for an even n and positive integer k and we conclude that graph $\Gamma_{\varphi_{n,k}}$ can be drawn in the plane arbitrarily close to $\widetilde{A}_i$.

We continue with some observations that are going to be important later in this document.

Recall that $G_n \subset A_i$ denotes the set $\widehat{aa_n} \cup \widehat{a_0 s_0}$.

Observation 4.11. The map $\varphi_{n,k}$ restricted to G_n^ν is the identity on G_n^ν. Additionally, G_n^ν is a component of $\varphi_{n,k}{}^{-1}(G_n^\nu)$.

Observation 4.12. Let L be an arc contained either in $\operatorname{int}\left(\widehat{a_0^\nu a_n^\nu}\right)$ or in $\operatorname{int}\left(\widehat{s_j^\nu b_j^\nu}\right)$ for some $j = 1 \dots, i$, and let K be a component of $\varphi_{n,k}{}^{-1}(L)$. Then K is either L or an arc contained in $\operatorname{int}\left(\widehat{s_{i+1}^\lambda b_{i+1}^\lambda}\right)$ where λ is either ν or $\tau(\nu)$. In both cases $\varphi_{n,k}$ restricted to K is a homeomorphism of K onto L.

Observation 4.13. Let $c^\nu \in G_n^\nu \setminus \{s_0^\nu\}$. Then the conclusion of the above observation is also true if $L = \widehat{c^\nu s_0^\nu}$ and K is a component of $\varphi_{n,k}{}^{-1}(L)$ such that $c^\nu \in \varphi_{n,k}(K)$.

Observation 4.14. $\varphi_{n,k}{}^{-1}\left(\widehat{a^\nu a_0^\nu} \setminus \{a_0^\nu\}\right) = \widehat{a^\nu a_0^\nu} \setminus \{a_0^\nu\}$.

Observation 4.15. If $\mu \in \{0,2\}$ then the following properties are true.

(1) $\varphi_{n,k}\big|_{\widehat{s^\mu_{i+1} \underline{b}^\mu_{i+1}}}$ does not depend on the value of k.
(2) $\varphi_{n,k}\left(\widehat{\underline{b}^\mu_{i+1} b^\mu_{i+1}}\right) = Y^{\tau(\mu)}$.
(3) $\varphi_{n,k}\left(\widehat{s^{\tau(\mu)}_{i+1} \underline{s}^{\tau(\mu)}_{i+1}}\right) = Y^{\tau(\mu)} \cup \widehat{y_0^{\tau(\mu)} s_{i+1}^{\tau(\mu)}}$.
(4) $\varphi_{n,k}\big|_{\widehat{\underline{s}^{\tau(\mu)}_{i+1} \underline{b}^{\tau(\mu)}_{i+1}}}$ does not depend on the value of k.
(5) $\varphi_{n,k}\left(\widehat{\underline{b}^{\tau(\mu)}_{i+1} b^{\tau(\mu)}_{i+1}}\right) = Y^{\mu}$.

4.3. **Preliminary definition of** X**.** In this section we give a preliminary definition of our example X in terms of an arbitrary sequence of positive integers $\Sigma = (k_0, k_1, \ldots)$. We use this definition to prove basic properties of X. For instance, we show that X can be embedded into the plane for all choices of Σ. Later, we will select by induction a specific sequence Σ that will allow us to prove that X admits a simple canal for every embedding into $\mathbb{R}^2$.

For each nonnegative integer n, set $i = \lceil n/2 \rceil$, $X_n = \widetilde{A}_i$ and $f_n = \varphi_{n,k_n}$. Define X to be $\varprojlim \{X_n, f_n\}_{n=0}^{\infty}$.

$$(*) \qquad X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} X_4 \xleftarrow{f_4} X_5 \xleftarrow{f_5} X_6 \xleftarrow{f_6} \cdots$$

$$\widetilde{A}_0 \xleftarrow{\varphi_{0,k_0}} \widetilde{A}_1 \xleftarrow{\varphi_{1,k_1}} \widetilde{A}_1 \xleftarrow{\varphi_{2,k_2}} \widetilde{A}_2 \xleftarrow{\varphi_{3,k_3}} \widetilde{A}_2 \xleftarrow{\varphi_{4,k_4}} \widetilde{A}_3 \xleftarrow{\varphi_{5,k_5}} \widetilde{A}_3 \xleftarrow{\varphi_{6,k_6}} \cdots$$

For all integers j and l such that $l > j \geq 0$ define $f_{jl} : X_l \to X_j$ as $f_{jl} = f_j \circ f_{j+1} \circ \cdots \circ f_{l-1}$. Additionally, define f_{jj} to be the identity on X_j. Let π_j denote the projection of X onto X_j.

Observation 4.16. $X_n \subset X_{n+1}$ for each nonnegative integer n.

We may assume that $\bigcup_{n=0}^{\infty} X_n$ is contained in the plane such that the diameter of $\bigcup_{n=0}^{\infty} X_n$ is ≤ 1. Let d denote the standard Euclidean metric in the plane. Let ρ_n denote d restricted to X_n. Let ρ denote the standard product metric on X defined by $\rho(x', x'') = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \rho(x'_n, x''_n)$ where $x' = (x'_0, x'_1, \ldots)$ and $x'' = (x''_0, x''_1, \ldots)$ are arbitrary points in X.

Proposition 4.17. *X_n is a tree for each nonnegative integer n. Thus, X is a tree-like continuum.*

In the rest of this section we argue that continuum X can be embedded in the plane. For a planar set Z we from now onwards denote by $\mathrm{cl}(Z)$ the *closure*, by $\mathrm{bd}(Z)$ the *boundary* and by $\mathrm{int}(Z)$ the *interior* of Z in the plane. If we write $\mathrm{int}\left(\widehat{c'c''}\right)$ for $\widehat{c'c''}$ being an arc, recall that we mean the open arc $\widehat{c'c''} \setminus \{c', c''\}$, since we do not specify in which space the arc $\widehat{c'c''}$ lies. It is going to be clear from the context which topology we mean when we use the notation for interior. Now say that Z is a planar tree and $\epsilon > 0$. The *thickened tree* Z^ϵ is the closure of the ϵ-neighbourhood of the tree Z. A map is a *near homeomorphism*, if it is a uniform limit of homeomorphisms. Recall that we observed in Subsection 4.1 and 4.2 that for every nonnegative integer n the graph Γ_{f_n} of the map $f_n : X_{n+1} \to X_n$ can be drawn in the plane arbitrarily close to X_n. That is equivalent to saying that Γ_{f_n} can be drawn in X_n^{ϵ} (and thus also in $\Gamma_{f_{n-1}}^{\epsilon}$) for an arbitrarily small $\epsilon > 0$. Therefore, for every $\epsilon > 0$ there exists $\epsilon' > 0$ such that $\Gamma_{f_n}^{\epsilon'} \subset \Gamma_{f_{n-1}}^{\epsilon}$ for every positive n. Let $\zeta_{\epsilon'}^{\epsilon} : \Gamma_{f_n}^{\epsilon'} \to \Gamma_{f_{n-1}}^{\epsilon}$ be an embedding of $\Gamma_{f_n}^{\epsilon'}$ into $\Gamma_{f_{n-1}}^{\epsilon}$. Fix a sequence $\epsilon_n \to 0$ so that $\Gamma_{f_n}^{\epsilon_n} \subset \Gamma_{f_{n-1}}^{\epsilon_{n-1}}$ as $n \to \infty$. Let us observe the inverse limit sequence $\varprojlim \left\{\Gamma_{f_n}^{\epsilon_n}, \zeta_{\epsilon^n}^{\epsilon^{n-1}}\right\}_{n=0}^{\infty}$. Since Γ_{f_n} is drawn (point-wise) arbitrarily close to X_n and $\epsilon_n \to 0$, the conditions of the Anderson-Choquet embedding theorem (Theorem 1 from [1]) are satisfied and thus $\varprojlim \left\{\Gamma_{f_n}^{\epsilon_n}, \zeta_{\epsilon^n}^{\epsilon^{n-1}}\right\}_{n=0}^{\infty}$ is homeomorphic to $\bigcap_{n=0}^{\infty} \mathrm{cl}\left(\bigcup_{j>n} \Gamma_j^{\epsilon_j}\right)$. Furthermore, $\bigcap_{n=0}^{\infty} \mathrm{cl}\left(\bigcup_{j>n} \Gamma_j^{\epsilon_j}\right) = \bigcap_{n=0}^{\infty} \zeta_{\epsilon^n}^{\epsilon^{n-1}} \circ \Gamma_n^{\epsilon_n}$ which is a nested intersection of planar continua and thus a planar continuum. Moreover, since there exists a near homeomorphism from space $\zeta_{\epsilon^n}^{\epsilon^{n-1}}\left(\Gamma_{f_n}^{\epsilon_n}\right)$ to the space $f_n(X_n)$ for every positive n, it follows by Theorem 3 from [7], that $\varprojlim \left\{\Gamma_{f_n}^{\epsilon_n}, \zeta_{\epsilon^n}^{\epsilon^{n-1}}\right\}_{n=0}^{\infty}$ is homeomorphic to $\varprojlim \{X_n, f_n\}_{n=0}^{\infty}$ and thus planar.

5. Auxiliary observations

In this section we state some propositions and observations which are going to be important later in the document when we prove that the rays R^{ν} are dense in X and have a free side.

The following observation is a simple consequence of 4.6 and 4.11.

Observation 5.1. Suppose that n is a nonnegative integer and $\nu \in \{0,1,2,3\}$. Then f_n restricted to G_n^{ν} is the identity on G_n^{ν}. Additionally, G_n^{ν} is a component of $f_n{}^{-1}\left(G_n^{\nu}\right)$.

Using the above observation and the inclusion $G_j^{\nu} \subset G_{j+1}^{\nu}$ repeatedly, we infer the following observation.

Observation 5.2. Suppose $\nu \in \{0,1,2,3\}$, and n and l are integers such that $0 \le n \le l$. Then f_{nl} restricted to G_n^{ν} is the identity on G_n^{ν}. Additionally, G_n^{ν} is a component of $f_{nl}{}^{-1}\left(G_n^{\nu}\right)$.

For each nonnegative integer n and each $\nu \in \{0,1,2,3\}$, let $S_n^{\nu} \subset X_n$ be defined by $S_0^{\nu} = \widehat{a_0^{\nu} y_0^{\nu}} \cup Y^{\nu}$ and $S_n^{\nu} = S_0^{\nu} \cup Y^{\nu} \cup \bigcup_{j=1}^{\lceil n/2 \rceil} \widehat{s_j^{\nu} b_j^{\nu}}$ if $n > 0$. Set $M_n^{\nu} = \widehat{a^{\nu} t_0^*} \cup S_n^{\nu}$.

Observation 5.3. $X_l = T^* \cup \bigcup_{\nu=0}^{3} M_l^{\nu}$ for all integers $l \ge 0$.

Observation 5.4. $M_l^{\nu} \subset f_l\left(M_{l+1}^{\nu}\right)$ for all integers $l \ge 0$ and $\nu \in \{0,1,2,3\}$.

Observation 5.5. $T^* \subset f_l\left(T^*\right)$ for all integers $l \ge 0$.

Observation 5.6. $T^* \subset f_l\left(\widehat{m_l^{\nu} a_{l+1}^{\nu}}\right)$ for all even integers $l \ge 0$ and $\nu \in \{0,1,2,3\}$.

Observation 5.7. $M_l^{\mu} \subset f_l\left(\widehat{m_l^{\nu} a_{l+1}^{\nu}}\right)$ where $l > 0$ is odd, $\mu = \mod ((l-1)/2, 4)$ and $\nu \in \{0,1,2,3\}$.

Proposition 5.8. *Suppose n is a nonnegative integer, $j \ge n+7$ is an even integer and $\nu \in \{0,1,2,3\}$. Then* $f_{nj}\left(\widehat{m_{j-1}^{\nu} a_j^{\nu}}\right) = X_n$.

Proof. Let $\mu_0 = \mod((l-2)/2, 4)$, $\mu_1 = \mod((l-4)/2, 4)$, $\mu_2 = \mod((l-6)/2, 4)$ and $\mu_3 = \mod((l-8)/2, 4)$ for some $l \geq 8$. Observe that $\{\mu_0, \mu_1, \mu_2, \mu_3\} = \{0,1,2,3\}$. It follows from 5.7 that $M^{\mu_0}_{j-1} \subset f_{j-1}\left(\widehat{m^{\nu}_{j-1} a^{\nu}_j}\right)$. Using 5.4 we infer

$$M^{\mu_0}_l \subset f_{lj}\left(\widehat{m^{\nu}_{j-1} a^{\nu}_j}\right) \text{ for } l \leq j-1. \tag{μ_0}$$

Since $T^* \subset f_{j-2}\left(M^{\mu_0}_l\right)$ by 5.6, it follows from 5.5 that

$$T^* \subset f_{lj}\left(\widehat{m^{\nu}_{j-1} a^{\nu}_j}\right) \text{ for } l \leq j-2. \tag{T}$$

It follows from 5.7 that $M^{\mu_1}_{j-3} \subset f_{j-3}\left(M^{\mu_0}_{j-2}\right)$. Using again 5.4 we infer

$$M^{\mu_1}_l \subset f_{lj}\left(\widehat{m^{\nu}_{j-1} a^{\nu}_j}\right) \text{ for } l \leq j-3. \tag{μ_1}$$

We infer the following two properties in a similar way.

$$M^{\mu_2}_l \subset f_{lj}\left(\widehat{m^{\nu}_{j-1} a^{\nu}_j}\right) \text{ for } l \leq j-5, \text{ and} \tag{μ_2}$$

$$M^{\mu_3}_l \subset f_{lj}\left(\widehat{m^{\nu}_{j-1} a^{\nu}_j}\right) \text{ for } l \leq j-7. \tag{μ_3}$$

We now complete the proof of the proposition by combining (μ_0), (T), (μ_1), (μ_2), (μ_3), and 5.3. □

Proposition 5.9. *Let n be a nonnegative integer, $\nu \in \{0,1,2,3\}$ and let $x \in X$ be such that $\pi_n(x) \in G^{\nu}_n \setminus \{s^{\nu}_0\}$. For each $l \geq n$ let K_l be the component of ${f_{nl}}^{-1}\left(\widehat{\pi_n(x)\, s^{\nu}_0}\right)$ containing $\pi_l(x)$. Then,*

(1) K_l is either $\widehat{\pi_n(x)\, s^{\nu}_0}$, or an arc contained in $\operatorname{int}\left(\widehat{a^{\lambda}_0 a^{\lambda}_l}\right)$ for some $\lambda \in \{0,1,2,3\}$, or an arc contained in $\operatorname{int}\left(\widehat{s^{\lambda}_j b^{\lambda}_j}\right)$ for some $\lambda \in \{0,1,2,3\}$ and $j \leq \lceil \frac{l}{2} \rceil$.
(2) $f_{l-1}|_{K_l}$ is a homeomorphism onto K_{l-1} for each $l > n$.

Proof. Observe that the proposition is trivial for $l = n$. Assume that the proposition is true for some $l \geq n$. To complete the proof it is enough to show that the proposition will be also true if l is replaced by $l+1$.

Since $f_l \circ \pi_{l+1}(x) = \pi_l(x) \in K_l$, it follows that $\pi_{l+1}(x) \in {f_l}^{-1}(K_l)$. Denote by J the component of ${f_l}^{-1}(K_l)$ containing $\pi_{l+1}(x)$. We will prove that $J = K_{l+1}$. Since $f_{n,l+1}(J) = f_{nl} \circ f_l(J) \subset f_{nl}(K_l) \subset \widehat{\pi_n(x)\, s^{\nu}_0}$, it follows that $J \subset K_{l+1}$. On the other hand, $\pi_l(x) \in f_l(K_{l+1}) \subset K_l$ and $\pi_{l+1}(x) \in K_{l+1}$, then $K_{l+1} \subset J$. Consequently $J = K_{l+1}$.

Using the inductive assumption, we may consider the following three cases.

(a) $K_l = \widehat{\pi_n(x)\, s^{\nu}_0}$,
(b) K_l is an arc contained contained either in $\operatorname{int}\left(\widehat{a^{\lambda}_0 a^{\lambda}_l}\right)$ or in $\operatorname{int}\left(\widehat{s^{\lambda}_j b^{\lambda}_j}\right)$ for some $\lambda \in \{0,1,2,3\}$ and $j \leq \lceil \frac{l}{2} \rceil$.

To complete the proof, it is enough to show conditions (1) and (2) from the statement of proposition with l replaced by $l+1$. Namely, we need to show:

(1′) $K_{l+1} = J$ is either $\widehat{\pi_n(x)\, s^{\nu}_0}$, or an arc contained in $\operatorname{int}\left(\widehat{a^{\lambda'}_0 a^{\lambda'}_{l+1}}\right)$ for some $\lambda' \in \{0,1,2,3\}$, or an arc contained in $\operatorname{int}\left(\widehat{s^{\lambda}_j b^{\lambda}_j}\right)$ for some $\lambda \in \{0,1,2,3\}$ and $j \leq \lceil \frac{l+1}{2} \rceil$.
(2′) $f_l|_{K_{l+1}}$ is a homeomorphism of $K_{l+1} = J$ onto K_l.

Recall that $f_l = \varphi_{l,k}$ where $k = k_l$ and $\varphi_{l,k}$ is described either in Subsection 4.1 if l is odd, or in Subsection 4.2 if l is even.

Case (a): $K_l = \widehat{\pi_n(x)\, s_0^\nu}$. In this case (1′) and (2′) follow from Observations 4.8 and 4.13 used with $n = l$, $c^\nu = \pi_n(x)$, $L = \widehat{\pi_n(x)\, s_0^\nu}$ and $K = K_{l+1} = J$.

Case (b): K_l is an arc contained contained either in $\operatorname{int}\left(\widehat{a_0^\lambda a_l^\lambda}\right)$ or in $\operatorname{int}\left(\widehat{s_j^\lambda b_j^\lambda}\right)$. In this case (1′) and (2′) follow from Observations 4.7 and 4.12 used with $n = l$, $\nu = \lambda$, $L = K_l$ and $K = K_{l+1} = J$. □

Corollary 5.10. *Let n be a nonnegative integer, $\nu \in \{0,1,2,3\}$ and let $x \in X$ be such that $\pi_n(x) \in G_n^\nu \setminus \{s_0^\nu\}$. Then there exists an arc $K \subset X$ such that x is an endpoint of K and $\pi_n|_K$ is a homeomorphism onto $\widehat{\pi_n(x) s_0^\nu}$.*

Proof. Let $K_n, K_{n+1}, K_{n+2}, \ldots$ be as in Proposition 5.9. Set $K_j = f_{jn}(K_n)$ for $j = 0, \ldots, n-1$. Let $K = \varprojlim \{K_l, f_l|_{K_l}\}_{l=0}^\infty$. Observe that K is an arc with the required properties. □

6. Sets F_n and the operation $[\cdot]$

This section will provide us with a set-up to work with specific subsets of the continuum X. Moreover, we will get some insight into the topological structure of X.

For all nonnegative integers n, let F_n denote the set of all points $z \in X_n$ with the property that $f_l(z) = z$ for all integers $l \geq n$.

The following observation is a simple consequence of (O-1) and (E-1).

Observation 6.1. Let $\nu \in \{0,1,2,3\}$ and let n be a positive integer. Then

(1) $\widehat{a^\nu a_0^\nu}$, T^*, Y^ν, $\widehat{a_0^\nu s^\nu}$ and G_0^ν are contained in F_0,
(2) $\widehat{a^\nu a_n^\nu}$, G_n^ν and $\bigcup_{j=1}^{\lceil n/2 \rceil} \widehat{s_j^\nu b_j^\nu}$ are contained in F_n.

For each $z \in F_n$, let $[z]_n$ denote the point $\left(f_{0n}(z), f_{1n}(z), \ldots, f_{n-1n}(z), z, z, \ldots\right) \in X$.

Observation 6.2. If n and l are integers such that $l \geq n$, and $z \in F_n$, then $z \in F_l$ and $[z]_n = [z]_l$.

For any $z \in \bigcup_{j=0}^\infty F_j$, set $[z] = [z]_n$ where n is any nonnegative integer such that $z \in F_n$. By the above observation, this definition does not depend on the choice on n.

Observation 6.3. For each $z \in F_n$, the definition of $[z]$ may depend on the choice of $k_0, \ldots, k_{n-1}$, but it does not depend of any choice of $k_n, k_{n+1}, \ldots$. Moreover, $[z] \in X$ regardless of how $k_n, k_{n+1}, \ldots$ are defined as long as the sequence Σ used in the definition of X (see Subsection 4.3) has the beginning $k_0, \ldots, k_{n-1}$.

Observation 6.4. If $z \in F_n$, then $\pi_l([z]) = z$ for all integers $l \geq n$.

For each set $Z \subset F_n$, let $[Z]$ denote the set $\{[z] \in X \mid z \in Z\}$.

Observation 6.5. If $Z \subset F_n$ then $\pi_l|_{[Z]}$ is a homeomorphism onto $Z \subset X_n \subset X_l$ for all $l \geq n$. Hence, $[Z] \subset X$ homeomorphic to Z. In particular, $[Z]$ is an arc if Z is an arc. Also, $[Z]$ is a simple triod if Z is is a simple triod.

Corollary 6.6. *$[T^*]$ is a simple triod. Also, $[Y^\nu]$ is a simple triod for each $\nu \in \{0,1,2,3\}$.*

The following proposition is a simple consequence of Observations 6.1, 6.3 and 6.5.

Proposition 6.7. *Let $\nu \in \{0,1,2,3\}$ and let n be a nonnegative integer. Then the arc $\widehat{a^\nu a_n^\nu} \subset F_n$ and the arc $\left[\widehat{a^\nu a_n^\nu}\right] \subset X$ does not depend on k_l for any $l \geq n$.*

Proposition 6.8. *Let n be an even nonnegative integer and let $i = n/2$. Then $\widehat{s^{\nu}_{i+1} b^{\nu}_{i+1}} \subset F_{n+1}$ for each $\nu \in \{0,1,2,3\}$. Moreover, if $\mu \in \{0,2\}$ then*

(1) $\left[\widehat{s^{\mu}_{i+1} b^{\mu}_{i+1}}\right] = \left[\widehat{s^{\mu}_{i+1} \underline{b}^{\mu}_{i+1}}\right] \cup \left[\widehat{\underline{b}^{\mu}_{i+1} b^{\mu}_{i+1}}\right]$,

(2) $\pi_n\left(\left[\widehat{\underline{b}^{\mu}_{i+1} b^{\mu}_{i+1}}\right]\right) = Y^{\tau(\mu)}$,

(3) $\left[\widehat{s^{\tau(\mu)}_{i+1} b^{\tau(\mu)}_{i+1}}\right] = \left[\widehat{s^{\tau(\mu)}_{i+1} \underline{s}^{\tau(\mu)}_{i+1}}\right] \cup \left[\widehat{\underline{s}^{\tau(\mu)}_{i+1} \underline{b}^{\tau(\mu)}_{i+1}}\right] \cup \left[\widehat{\underline{b}^{\tau(\mu)}_{i+1} b^{\tau(\mu)}_{i+1}}\right]$,

(4) $\pi_n\left(\left[\widehat{s^{\tau(\mu)}_{i+1} \underline{s}^{\tau(\mu)}_{i+1}}\right]\right) = Y^{\tau(\mu)} \cup \widehat{y^{\tau(\mu)}_0 s^{\tau(\mu)}_{i+1}}$,

(5) $\pi_n\left(\left[\widehat{\underline{b}^{\tau(\mu)}_{i+1} b^{\tau(\mu)}_{i+1}}\right]\right) = Y^{\mu}$, *and*

(6) the arcs $\left[\widehat{s^{\mu}_{i+1} \underline{b}^{\mu}_{i+1}}\right]$ *and* $\left[\widehat{\underline{s}^{\tau(\mu)}_{i+1} \underline{b}^{\tau(\mu)}_{i+1}}\right]$ *do not depend on k_l for any $l \geq n$.*

Proof. $\widehat{s^{\nu}_{i+1} b^{\nu}_{i+1}} \subset F_{n+1}$ by Observation 6.1, so the arcs $\left[\widehat{s^{\mu}_{i+1} b^{\mu}_{i+1}}\right]$ and $\left[\widehat{s^{\tau(\mu)}_{i+1} b^{\tau(\mu)}_{i+1}}\right]$ are well defined. Claims (1) and (3) follow from the choice of $\underline{b}^{\mu}_{i+1}$, $\underline{b}^{\tau(\mu)}_{i+1}$ and $\underline{s}^{\tau(\mu)}_{i+1}$ presented in the beginning of Subsection 4.2.

To prove Claim (2) notice that $\pi_{n+1}\left(\left[\widehat{\underline{b}^{\mu}_{i+1} b^{\mu}_{i+1}}\right]\right) = \widehat{\underline{b}^{\mu}_{i+1} b^{\mu}_{i+1}}$ the Observation 6.4. Since $\pi_n = f_n \circ \pi_{n+1}$ and $f_n = \varphi_{n,k}$, Claim (2) follows now from Observation 4.15(2). Proofs of Claims (4) and (5) are essentially the same except that we use Observation 4.15 parts (3) and (5).

To complete the proof of the proposition notice that Claim (6) follows from Observation 6.3 and parts (1) and (4) of Observation 4.15. □

Set

$$\mathscr{L}_{i+1} = \left\{\left[\widehat{s^{0}_{i+1} \underline{b}^{0}_{i+1}}\right], \left[\widehat{\underline{s}^{1}_{i+1} \underline{b}^{1}_{i+1}}\right], \left[\widehat{s^{2}_{i+1} \underline{b}^{2}_{i+1}}\right], \left[\widehat{\underline{s}^{3}_{i+1} \underline{b}^{3}_{i+1}}\right]\right\}.$$

The following corollary is a restatement of part (6) of Proposition 6.8.

Corollary 6.9. *All elements of $\mathscr{L}_{i+1}$ do not depend on k_l for any $l \geq n$.*

7. Basic properties of rays R^{ν} and spurs S^{ν}

In this section we will first show that X contains four dense rays, each of which must have a free fully accessible side under every embedding into $\mathbb{R}^2$. At the end of the section we will observe that there are four spurs attached on a non-free side of four dense rays. The spurs are going to be important later in the document when we construct a simple dense canal in every planar embedding of X.

Since $\left[\widehat{a^{\nu} a^{\nu}_0}\right] \subset \left[\widehat{a^{\nu} a^{\nu}_1}\right] \subset \left[\widehat{a^{\nu} a^{\nu}_2}\right] \subset \ldots$ and each $\left[\widehat{a^{\nu} a^{\nu}_l}\right]$ is an arc, it follows that

$$R^{\nu} = \bigcup_{l \geq 0} \left[\widehat{a^{\nu} a^{\nu}_l}\right] \text{ is a ray in } X.$$

Proposition 7.1. *R^{ν} is dense in X for each $\nu \in \{0,1,2,3\}$.*

Proof. To prove the proposition it is enough to show that $\pi_n(R^{\nu}) = X_n$ for each nonnegative integer n. Take an even integer $j \geq n+7$. Since $\widehat{m^{\nu}_{j-1} a^{\nu}_j} \subset F_j$ by 6.1(2), it follows from 6.5 that $\pi_j\left(\left[\widehat{m^{\nu}_{j-1} a^{\nu}_j}\right]\right) = \widehat{m^{\nu}_{j-1} a^{\nu}_j}$. Since $f_{nj}\left(\widehat{m^{\nu}_{j-1} a^{\nu}_j}\right) = X_n$ by 5.8, $f_{nj} \circ \pi_j = \pi_n$ and $\left[\widehat{m^{\nu}_{j-1} a^{\nu}_j}\right] \subset R^{\nu}$, we get that $\pi_n(R^{\nu}) = X_n$. □

Proposition 7.2. *Suppose n is a positive integer, $z \in \widehat{a_n a_{n+1}}$ and $\lambda, \nu \in \{0,1,2,3\}$. Then $f_{n-1} \circ f_n\left(z^\lambda\right) = f_{n-1} \circ f_n\left(z^\nu\right)$.*

Proof. We will consider the following two cases: $z \in \widehat{a_n m_n}$ and $z \in \widehat{m_n a_{n+1}}$.
Case $z \in \widehat{a_n m_n}$. By using either (O-4) if n is odd, or (E-2) if n is even, we get the same result that $f_n(z^\lambda) = w^\lambda$ and $f_n(z^\nu) = w^\nu$ where $w = \beta_n(z)$. Since $\beta_n = \alpha\left\langle (a_n, t_0), \widehat{a_n m_n}, A_0 \right\rangle$, the point w belongs to $\widehat{a_n t_0}$. Now, by using either (O-6) for $n-1$ if $n-1$ is odd, or (E-4) for $n-1$ if $n-1$ is even, we get the same result that $f_{n-1}(w^\lambda) = t_0^* = f_{n-1}(z^\nu)$. So, the proposition is true in this case.
Case $z \in \widehat{m_n a_{n+1}}$. If n is odd then $f_n\left(z^\lambda\right) = \gamma_{\mathrm{o}}(z) = f_n\left(z^\nu\right)$ by (O-5). If n is even then $f_n\left(z^\lambda\right) = \gamma_{\mathrm{e}}(z) = f_n\left(z^\nu\right)$ by (E-3). So, $f_n\left(z^\lambda\right) = f_n\left(z^\nu\right)$ regardless whether n is odd or even. Consequently, the proposition is true. □

Corollary 7.3. *For each $\nu \in \{0,1,2,3\}$, let $\psi^\nu : \widehat{a t_0} \setminus \{t_0\} \to R^\nu$ be the function defined by $\psi^\nu(z) = [z^\nu]$. Then ψ^ν is a continuous injection of $\widehat{a t_0} \setminus \{t_0\}$ onto R^ν. Furthermore, if $\lambda, \nu \in \{0,1,2,3\}$ then $\lim \rho\left([z^\nu], \left[z^\lambda\right]\right) = 0$ as $z \in \widehat{a t_0} \setminus \{t_0\}$ converges to t_0.*

Proposition 7.4. *Let n be a nonnegative integer and let $\nu \in \{0,1,2,3\}$. Suppose $C \subset X$ is a connected set such that $C \cap \left[G_n^\nu\right] \neq \emptyset$ and $\pi_n(C) \subset G_n^\nu$. Then $C \subset \left[G_n^\nu\right]$.*

Proof. Let $c \in G_n^\nu$ be such that $[c] \in C$. Let l be an arbitrary integer greater than n. The set $\pi_l(C)$ is connected since C is connected. Observe that G_n^ν is a component of ${f_{nl}}^{-1}\left(G_n^\nu\right)$ by 5.1. Since $\pi_n(C) \subset G_n^\nu$ and $\pi_n = f_{nl} \circ \pi_l$, the connected set $\pi_l(C)$ is contained in the component of ${f_{nl}}^{-1}\left(G_n^\nu\right)$ containing $\pi_l([c]) = c$. Thus, $\pi_l(C) \subset G_n^\nu$ for all integers $l \geq n$. It follows that $C \subset \left[G_n^\nu\right]$. □

Proposition 7.5. *Let n be a nonnegative integer and let $\nu \in \{0,1,2,3\}$. Suppose L is an arc contained in X such that $L \cap \left[\widehat{a^\nu a_n^\nu} \setminus \left\{a_n^\nu\right\}\right]$ consists of a single point e which is an endpoint of L. Then $e = \left[a_0^\nu\right]$ and one of the arcs L and $\left[\widehat{a_0^\nu s_0^\nu}\right]$ must contain the other.*

Proof. Let e' denote the other endpoint of L. Suppose that $(L \setminus \{e\}) \cap \left[G_n^\nu\right] = \emptyset$. In that case, since $\pi_n(e)$ belongs to the interior of G_n^ν in X_n, there is an arc C such that $e \in C \subset L$ such that $\pi_n(C) \subset G_n^\nu$. Since $e \in \left[G_n^\nu\right]$ it follows by 7.4 that $C \subset \left[G_n^\nu\right]$, which is a contradiction. So, $L \cap \left[G_n^\nu\right]$ is nondegenerate. Since X is tree-like, $L \cap \left[G_n^\nu\right]$ is an arc contained in $\left[\widehat{a_0^\nu s_0^\nu}\right]$. It follows that $e = \left[a_0^\nu\right]$. Denote by u the other end of the arc $L \cap \left[G_n^\nu\right]$. If $u = e'$, then $L \subset \left[\widehat{a_0^\nu s_0^\nu}\right]$ and the proposition would be true. So, we may assume that $u \neq e'$. If $u = \left[s_0^\nu\right]$, then $\left[\widehat{a_0^\nu s_0^\nu}\right] \subset L$ and again the proposition would be true. So, we may assume that $u \in \operatorname{int}\left(\left[\widehat{a_0^\nu s_0^\nu}\right]\right)$. Since $\pi_n(u) \in \operatorname{int}\left(\widehat{a_0^\nu s_0^\nu}\right)$ and $\operatorname{int}\left(\widehat{a_0^\nu s_0^\nu}\right)$ is open in X_n, there is a point $z \in \operatorname{int}\left(\widehat{u e'}\right)$ such that $\pi_i\left(\widehat{uz}\right) \subset \widehat{a_0^\nu s_0^\nu} \subset G_n^\nu$. Since $u \in \left[G_n^\nu\right] \cap \widehat{uz}$, it follows from 7.4 that $\widehat{uz} \subset \left[G_n^\nu\right]$. This last contradiction completes the proof of 7.5. □

Proposition 7.6. *Suppose $\mathscr{B}$ is an open covering of the open interval $(-1,1)$ and let $v \in (-1,1)$. Then there are sequences $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ such that*

(1) $v_1 = v > v_2 > u_1 > v_3 > u_2 > v_4 > u_3 > v_5 > u_4 > \ldots$,
(2) $\lim_{i\to\infty} u_i = \lim_{i\to\infty} v_i = -1$, *and*
(3) for each positive integer i there is $B_i \in \mathscr{B}$ such that $(u_i, v_i) \subset B_i$.

Lemma 7.7. *Let $h : X \to \mathbb{R}^2$ be an embedding and let $\nu \in \{0,1,2,3\}$. Then, for each $z \in R^\nu \setminus [a^\nu]$ there is a topological disk $D \subset \mathbb{R}^2$ such that $D \cap h(X) = h\left(\widehat{[a^\nu]\, z}\right)$.*

Proof. Let $z \in R^{\nu} \setminus [a^{\nu}]$. There is a positive integer l such that $z \in \left[\widehat{a^{\nu} a_l^{\nu}} \setminus \{a_l^{\nu}\}\right]$. Let I_1 denote the straight linear segment in $\mathbb{R}^2$ joining $(-1,0)$ and $(1,0)$. For each real number r such that $0 \le r \le 1$, let

$$H(r) = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid -1 + r \le x_1 \le 1 - r\right\}.$$

Let g be a homeomorphism of $\mathbb{R}^2$ onto itself such that

- $g \circ h([a^{\nu}]) = (-1,0)$, $g \circ h\left([a_0^{\nu}]\right) = (0,0)$, $g \circ h\left([a_l^{\nu}]\right) = (1,0)$, and
- $g \circ h\left(\left[\widehat{a^{\nu} a_l^{\nu}}\right]\right) = I_1$.

Let $\tilde{h}$ denote the composition $g \circ h$. We may assume without loss of generality that there exists $c \in \operatorname{int}\left(\widehat{a_0^{\nu} s_0^{\nu}}\right)$ such that $\tilde{h}\left(\left[\widehat{a_0^{\nu} c}\right]\right) \subset H(1/3)$ and $\tilde{h}\left(\left[\widehat{a_0^{\nu} c} \setminus \{[a_0^{\nu}]\}\right]\right)$ lies below I_1.

For each number r such that $-1 \le r \le 1$ and each positive ϵ, let $B(\epsilon, r)$ denote the set of all points $(x_1, x_2) \in \mathbb{R}^2$ such that $r - \epsilon \le x_1 \le r + \epsilon$ and $0 \le x_2 \le \epsilon$.

Claim 7.8. For each r such that $-1 < r < 1$ there exists a positive number ϵ such that

$$(B(\epsilon, r) \setminus I_1) \cap \tilde{h}(X) = \emptyset.$$

Proof of 7.8. Take an arbitrary r such that $-1 < r < 1$. Let η be the minimum of $r + 1$, $1 - r$ and the distance between $\tilde{h}([c])$ and I_1. Clearly $0 < \eta \le 1$ and $(r,0) \in H(\eta)$. There is a positive number δ such that for all $x', x'' \in X$ the following implication holds:

$$\varrho(x', x'') < \delta \quad \Rightarrow \quad d\left(\tilde{h}(x'), \tilde{h}(x'')\right) < \eta/3 \tag{1}$$

Let $n > l$ be such that for all $x', x'' \in X$ the following implication is true:

$$\pi_n(x') = \pi_n(x'') \quad \Rightarrow \quad \varrho(x', x'') < \delta \tag{2}$$

Let $U = \widehat{a^{\nu} a_l^{\nu}} \cup \widehat{a_0^{\nu} c} \setminus \{a^{\nu}, a_l^{\nu}, c\}$. Since $n > l$, U is contained in X_n. Observe that U is open in X_n. Since $\tilde{h}^{-1}((r,0)) \in \operatorname{int}\left(\left[\widehat{a^{\nu} a_l^{\nu}}\right]\right)$, it follows that

$$\pi_n \circ \tilde{h}^{-1}((r,0)) \in \operatorname{int}\left(\widehat{a^{\nu} a_l^{\nu}}\right) \subset U \tag{3}$$

Since $(r,0) \in B(\epsilon, r)$, U is open in X_n and $\pi_n \circ \tilde{h}^{-1}$ is continuous on $\tilde{h}(X)$, it follows from (3) that there is a positive number $\epsilon < \eta/3$ such that

$$\pi_n \circ \tilde{h}^{-1}\left(B(\epsilon, r) \cap \tilde{h}(X)\right) \subset U \tag{4}$$

Observe that $B(\epsilon, r) \subset H(2\eta/3)$ because $(r,0) \in H(\eta)$ and $\epsilon < \eta/3$.

We will now prove that ϵ satisfies the claim. Suppose to the contrary that there is a point $x \in X$ such that $\tilde{h}(x) \in B(\epsilon, r) \setminus I_1$. It follows from (4) that $\pi_n(x) \in U$. Corollary 5.10 implies that there exists an arc $K \subset X$ such that x is an endpoint of K and $\pi_n|_K$ is a homeomorphism onto $\widehat{\pi_n(x) s_1^{\nu}}$. There is a point $\hat{c} \in K$ such that $\pi_n(\hat{c}) = c$. Let $\hat{K}$ be the subarc of K with endpoints in x and $\hat{c}$. Observe that $\pi_n|_{\hat{K}}$ is a homeomorphism onto $\widehat{\pi_n(x) c}$.

Since $\pi_n(x) \in U \subset \operatorname{cl}(U) = \widehat{a^{\nu} a_l^{\nu}} \cup \widehat{a_0^{\nu} c}$, the arc $\widehat{\pi_n(x) c}$ is contained in $\widehat{a^{\nu} a_l^{\nu}} \cup \widehat{a_0^{\nu} c}$. It follows that $\widehat{\pi_n(x) c} \subset F_n$. Observe that $\widehat{\pi_n(x) c} \subset \widehat{a_0^{\nu} c}$ if $\pi_n(x) \in \widehat{a_0^{\nu} c}$, and $\widehat{\pi_n(x) c} = \widehat{\pi_n(x) a_0^{\nu}} \cup \widehat{a_0^{\nu} c}$ if $\pi_n(x) \in \widehat{a^{\nu} a_l^{\nu}} \setminus \{a_0^{\nu}\}$. Consequently, we have the following two cases.

(C-1) Either $\pi_n(x) \in \widehat{a_0^{\nu} c}$ and $\tilde{h}\left(\left[\widehat{\pi_n(x) c}\right]\right) \subset \tilde{h}\left(\left[\widehat{a_0^{\nu} c}\right]\right)$, or

(C-2) $\pi_n(x) \in \widehat{a^{\nu} a_l^{\nu}} \setminus \{a_0^{\nu}\}$ and $\tilde{h}\left(\left[\widehat{\pi_n(x) c}\right]\right) = \tilde{h}\left(\left[\widehat{\pi_n(x) a_0^{\nu}}\right]\right) \cup \tilde{h}\left(\left[\widehat{a_0^{\nu} c}\right]\right)$.

Let w be an arbitrary point in $\hat{K}$. Observe that $\pi_n(w) = \pi_n([\pi_n(w)])$. Using (2) we infer that $\varrho(w, [\pi_n(w)]) < \delta$. Now, using (1), we get the following result.

$$d\left(\tilde{h}(w), \tilde{h}([\pi_n(w)])\right) < \eta/3 \quad \text{for all } w \in \hat{K}. \tag{5}$$

In particular, $d\left(\tilde{h}(x), \tilde{h}([\pi_n(x)])\right) < \eta/3$. Since $\tilde{h}(x) \in B(\epsilon, r) \subset H\left(2\eta/3\right)$, $\tilde{h}([\pi_n(x)])$ must belong to $H\left(\eta/3\right)$. In the case (C-2) the arc $\tilde{h}\left(\left[\widehat{\pi_n(x) a_0^\nu}\right]\right)$ is contained in I_1 and its endpoints $\tilde{h}([\pi_n(x)])$ and $\tilde{h}\left(\left[a_0^\nu\right]\right) = (0,0)$ are both contained in $H(\eta/3)$. So, it that case, the arc $\tilde{h}\left(\left[\widehat{\pi_n(x) a_0^\nu}\right]\right) \subset H(\eta/3)$.

Observe that $\tilde{h}\left(\left[\widehat{a_0^\nu c}\right]\right) \subset H(\eta/3)$ because $\tilde{h}\left(\left[\widehat{a_0^\nu c}\right]\right) \subset H(1/3)$ and $\eta \le 1$. Therefore, $\tilde{h}\left(\left[\widehat{\pi_n(x) c}\right]\right) \subset H(\eta/3)$ in both cases (C-1) and (C-2). Now, (5) implies that

$$\tilde{h}\left(\hat{K}\right) \subset H(0) \tag{6}$$

Since $\pi_n(\hat{c}) = c$, we may infer from (5) that $d\left(\tilde{h}(\hat{c}), \tilde{h}([c])\right) < \eta/3$. Since $\tilde{h}([c])$ lies below I_1 and the distance between $\tilde{h}([c])$ is at least η, the point $\tilde{h}(\hat{c})$ also lies below I_1. Since $\tilde{h}(x) \in B(\epsilon, r) \setminus I_1$ lies above I_1, (6) implies that $\tilde{h}\left(\hat{K}\right) \cap I_1 \neq \emptyset$. Hence, $\hat{K} \cap \left[\widehat{a^\nu a_l^\nu}\right] \neq \emptyset$. Let e be the first point in the arc $\hat{K}$ oriented from x to $\hat{c}$ such that $e \in \left[\widehat{a^\nu a_l^\nu}\right]$ and let L be the subarc of $\hat{K}$ with endpoints x and e. This choice of L and e contradicts Proposition 7.5 since neither of L and $\left[\widehat{a_0^\nu s_0^\nu}\right]$ contains the other. So, Claim 7.8 is true. □

Let v be the first coordinate of $\tilde{h}(z)$. So, $\tilde{h}(z) = (v, 0)$. Let $\mathscr{B}$ be the collection of all open intervals in the form $(r-\epsilon, r+\epsilon)$ where r and ϵ are real numbers such that $-1 < r < 1$ and $\epsilon > 0$ and $(B(\epsilon, r) \setminus I_1) \cap \tilde{h}(X) = \emptyset$. It follows from the claim that $\mathscr{B}$ is a covering of the interval $(-1,1)$. Now, use Proposition 7.6 to get sequences $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ satisfying conditions (1)-(3) of the proposition, where condition (3) can be rephrased in the following way: for each nonnegative integer i there is $r_i \in (-1,1)$ and ϵ_i such that $r_i - \epsilon_i \le u_i < v_i \le r_i + \epsilon_i$ and $(B(\epsilon_i, r_i) \setminus I_1) \cap \tilde{h}(X) = \emptyset$. Denote by σ_i the minimum of $2^{-i}, \epsilon_1, \epsilon_2, \ldots, \epsilon_i$. Let P be the union of straight linear arcs joining the following sequence of consecutive points in $\mathbb{R}^2$:

$$(v,0) = (v_1, 0), (v_1, \sigma_1), (v_2, \sigma_1), (v_2, \sigma_2), (v_3, \sigma_2), (v_3, \sigma_3), (v_4, \sigma_3), \ldots$$

Observe that $\mathrm{cl}(P)$ is an arc intersecting $\tilde{h}\left(\left[\widehat{a^\nu z}\right]\right) \subset I_1$ only at the common endpoints $\tilde{h}([a^\nu])$ and $\tilde{h}(z)$. Thus, $\mathrm{cl}(P) \cup \tilde{h}\left(\left[\widehat{a^\nu z}\right]\right)$ is a simple closed curve bounding a disk which we denote by $\tilde{D}$. By our construction, $\left(\tilde{D} \setminus I_1\right) \cap \tilde{h}(X) = \emptyset$. Finally, set $D = g^{-1}\left(\tilde{D}\right)$ and observe that so defined D satisfies the conclusion of the lemma. □

Recall that a point p in a subset K of the plane is accessible from the complement of K provided there is an arc $L \subset \mathbb{R}^2$ such that $K \cap L = \{p\}$.

Proposition 7.9. *Let $h : X \to \mathbb{R}^2$ be an embedding and $\nu \in \{0,1,2,3\}$. Then for every $z \in R^\nu$ the point $h(z) \in h(R^\nu)$ is accessible from the complement of $h(X)$.*

Proof. Let $z \in R^\nu \setminus [a^\nu]$. By Lemma 7.7 there exists a topological disk $D \subset \mathbb{R}^2$ such that $D \cap h(X) = h\left(\widehat{[a^\nu]\, z}\right)$. Denote the arc $A = \mathrm{bd}(D) \setminus \mathrm{int}\left(h\left(\widehat{[a^\nu]\, z}\right)\right) \subset \mathbb{R}^2$. Let $u \in \mathrm{int}(A)$. Let L_0 be the subarc of A with endpoints u and $h([a^\nu])$. Clearly, $L_0 \cap h(X) = \left\{h([a^\nu])\right\}$. Similarly, if L_1 denotes the subarc of A with endpoints u and $h(z)$ then $L_0 \cap h(X) = \{h(z)\}$. It follows that both points $h([a^\nu])$ and $h(z)$ are accessible from the complement of $h(X)$. □

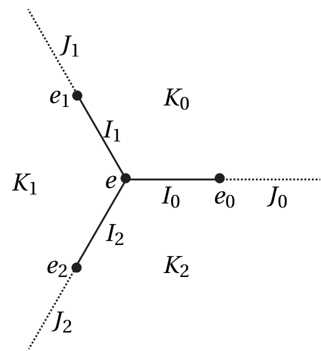


FIGURE 7. E

Recall that $S_0^v = \widehat{a_0^v y_0^v} \cup Y^v$. Since $S_0^v \subset A_0$ and $A_0 \subset A_i$ for all nonnegative integers i, $S_0^v \subset \widetilde{A}_i$ for all $v \in \{0,1,2,3\}$ and all nonnegative integers i. Consequently, $S_0^v \subset X_n$ for each nonnegative integer n. Using conditions (O-1) and (O-3) in case of odd n, and (E-1) and (E-5) in case of n even, we observe that f_n maps S_0^v onto itself. By the *spur* S^v we understand the subcontinuum of X defined by

$$S^v = \left\{x \in X \mid \pi_n(x) \in S_0^v \quad \text{for all nonnegative integer } n\right\}.$$

Observation 7.10. The following properties are true.

(1) S^v is a continuum containing the simple triod $[Y^v]$.
(2) $S^v \setminus [Y^v]$ is a ray converging to $[Y^v]$.
(3) $S^v \cap R^v = \left[a_0^v\right]$
(4) $S^v \cap [T^*] = \emptyset$.
(5) The four spurs are mutually disjoint.

8. ARCS WINDING AROUND A SIMPLE TRIOD

In this section we establish a language to describe winding of arcs from X around four simple triods $[Y^v]$ and $[T^*]$. This language will be used in the crux of proving of the existence of a simple dense canal in every planar embedding of X.

Throughout this section we use the following notation. If j is an integer then by $j_{(=3)}$, $j_{(+3)}$ and $j_{(-3)}$ we understand $\bmod (j,3)$, $\bmod (j+1,3)$ and $\bmod (j-1,3)$, respectively.

Let e be an arbitrary point in $\mathbb{R}^2$. Set $e_0 = e + (1,0)$, $e_1 = e + (\cos(2\pi/3), \sin(2\pi/3))$ and $e_2 = e + (\cos(4\pi/3), \sin(4\pi/3))$. For each $i \in \{0,1,2\}$, let I_i denote the straight linear segment in the plane joining e with e_i. Let H_i denote the half line with the endpoint e such that $I_i \subset H_i$. Set $J_i = H_i \setminus I_i$, $E = I_0 \cup I_1 \cup I_2$ and $K = \mathbb{R}^2 \setminus E$. Observe that $J_0 \cup J_1 \cup J_2$ separates K into three components whose closures in K can be enumerated K_0, K_1 and K_2 so that $K_0 \cap K_1 = J_1$, $K_1 \cap K_2 = J_2$ and $K_2 \cap K_0 = J_0$, see Figure 7.

Observation 8.1. $d(e_i, z) > 1$ for every $i \in \{0,1,2\}$ and $z \in K_{i_{(+3)}}$.

Observation 8.2. For every $i \in \{0,1,2\}$, the boundary of $K_{i_{(+3)}}$ in K is equal to $J_{i_{(-3)}} \cup J_{i_{(+3)}} = J_0 \cup J_1 \cup J_2 \setminus J_i$.

Observation 8.3. Let $i, j \in \{0,1,2\}$ be such that $i \neq j$. Then $d(w,z) > 1$ for all $w \in I_i$ and $z \in J_j$.

Observation 8.4. Let $i, j \in \{0,1,2\}$ be such that $i \neq j$. Then $d(w,z) > 1$ for all $w \in J_i$ and $z \in J_j$.

Lemma 8.5. *Let k be a positive integer, let $\widehat{c'c''}$ be an oriented arc, and let $c_0 = c', c_1, \ldots, c_{3k-1}, c_{3k} = c''$ be a strictly increasing sequence of points from $\widehat{c'c''}$. Suppose that $\alpha : \widehat{c'c''} \to E$ is such that*

(a) $\alpha(c_j) = e_{j_{(=3)}}$ for each $j = 0, \ldots, 3k$, and

(b) $\alpha|_{\widehat{c_j c_{j+1}}}$ is a homeomorphism onto $\alpha\widehat{(c_j)\,\alpha(c_{j+1}})\subset E$ for each $j = 0, \ldots, 3k-1$.

Denote by u_j the only point in $\widehat{c_j c_{j+1}}$ such that $\alpha(u_j) = e$. Finally, suppose $g : \widehat{c'c''} \to K$ is a mapping such that $d(g(z), \alpha(z)) \le 1$ for each $z \in \widehat{c'c''}$. Then for each $j = 0, \ldots, 3k-1$

(1)$_j$ $g(u_j) \in \operatorname{int}(K_{j_{(=3)}})$,

(2)$_j$ $g(z) \notin K_{j_{(+3)}}$ for all $z \in \widehat{c_j u_j}$, and

(3)$_j$ $g(z) \notin K_{j_{(-3)}}$ for all $z \in \widehat{u_j c_{j+1}}$.

Remark 8.6. Observe that α defined by conditions (a) and (b) in the above lemma has the same properties as $\alpha\langle W, \widehat{c'c''}, E\rangle$ defined in Definition 3.2 for $W = (e_0, e_1, e_2)^k \oplus e_0$.

Proof of 8.5. Let j be an arbitrary integer such that $0 \le j < 3k$. Since $\alpha(c_j) = e_{j_{(=3)}}$ and $\operatorname{mod}(j_{(=3)} + 1, 3) = j_{(+3)}$, it follows from Observation 8.1 used with $i = j_{(=3)}$, that $g(c_j) \notin K_{j_{(+3)}}$. Since $\alpha(\widehat{c_j u_j}) = I_{j_{(=3)}}$, it follows from observations 8.2 and 8.3, both used with $i = j_{(=3)}$, that $g(z) \notin K_{j_{(+3)}}$ for all $z \in \widehat{c_j u_j}$. So, the condition $(2)_j$ is true.

Since $\alpha(c_{j+1}) = e_{j_{(+3)}}$ and $\operatorname{mod}(j_{(+3)} + 1, 3) = j_{(-3)}$, it follows from Observation 8.1 used with $i = j_{(+3)}$, that $g(c_{j+1}) \notin K_{j_{(-3)}}$. Since $\alpha(\widehat{c_{j+1} u_j}) = I_{j_{(+3)}}$, it follows from observations 8.2 and 8.3, both used with $i = j_{(+3)}$, that $g(z) \notin K_{j_{(-3)}}$ for all $z \in \widehat{c_{j+1} u_j}$. So, the condition $(3)_j$ is true.

Since u_j belongs to both $\widehat{c_j u_j}$ and $\widehat{c_{j+1} u_j}$, the condition $(1)_j$ is a simple consequence of $(2)_j$ and $(3)_j$. Hence, the lemma is true. □

Let S^1 denote the unit circle in $\mathbb{R}^2$. For any two not antipodal points $e', e'' \in S^1$, let $\widehat{e'e''}$ denote the shorter of the two arcs contained in S^1 with their endpoints $e'$and e''. Let D denote the round disk in the plane with center e and radius 2. Observe that $K = \mathbb{R}^2 \setminus E$ is homeomorphic to $S^1 \times (0,\infty)$. Moreover, there exists a homeomorphism ω mapping $S^1 \times (0,\infty)$ onto K such that $\omega(S^1 \times (0,1]) = D \setminus E$, $\omega(\widehat{e_0 e_1} \times (0,\infty)) = K_0$, $\omega(\widehat{e_1 e_2} \times (0,\infty)) = K_1$, $\omega(\widehat{e_2 e_0} \times (0,\infty)) = K_2$ and, for each $i = \{0,1,2\}$ and each sequence $s = (z_j)_{j=1}^{\infty}$ of points in K_i, s converges in $\mathbb{R}^2$ if and only if the sequence $(\omega^{-1}(z_j))_{j=1}^{\infty}$ converges in $S^1 \times [0,\infty)$. Observe that $\omega(\{e_i\} \times (0,\infty)) = J_i$ for each $i = \{0,1,2\}$.

Let $\theta : \mathbb{R} \to S^1$ be defined by $(\cos(2\pi z), \sin(2\pi z))$ where $z \in \mathbb{R}$. Set $\widetilde{K} = \mathbb{R} \times (0,\infty)$, and let r_1 and r_2 denote the projections of $\widetilde{K}$ onto $\mathbb{R}$ and $(0,\infty)$, respectively. Let $p : \widetilde{K} \to K$ be the mapping defined by the formula $p(\tilde{z}) = \omega(\theta \circ r_1(\tilde{z}), r_2(\tilde{z}))$ where $\tilde{z} \in \widetilde{K}$.

Observation 8.7. p is a covering projection. p is periodic (with period 1) in the following sense $p(x_1 + 1, x_2) = p(x_1, x_2)$ for all $x_1 \in \mathbb{R}$ and $x_2 \in (0,\infty)$.

Observation 8.8. ω restricted to $S^1 \times \{1\}$ is a homeomorphism onto $\operatorname{bd}(D)$. p restricted to $\mathbb{R} \times \{1\}$ is a covering projection onto the simple closed curve $\operatorname{bd}(D)$.

Observation 8.9. Let $n \in \mathbb{Z}$ and let $\xi_n : \widetilde{K} \to \widetilde{K}$ be defined by $\xi_n(\tilde{z}) = (r_1(\tilde{z}) + n, r_2(\tilde{z}))$ where $\tilde{z} \in \widetilde{K}$. Then ξ_n is a well-defined homeomorphism of $\widetilde{K}$ onto itself such that $p \circ \xi_n = p$. Moreover, for all $\tilde{z}_0, \tilde{z}_1 \in \widetilde{K}$ such that $p(\tilde{z}_0) = p(\tilde{z}_1)$ there exists exactly one $n \in \mathbb{Z}$ such that $\xi_n(\tilde{z}_0) = \tilde{z}_1$.

Recall that, for each g which is a continuous mapping of a space Z into K, a continuous mapping $\tilde{g} : Z \to \widetilde{K}$ is called a lifting of g if $p \circ \tilde{g} = g$. If Z is connected then any two liftings of g are the same if they agree on one point; see [9, 1.34]. Also recall that if Z is simply connected and locally path connected, $g : Z \to K$ is continuous, $z_0 \in Z$ and $\tilde{z}_0 \in \widetilde{K}$ are such that $p(\tilde{z}_0) = g(z_0)$, then there is a lifting $\tilde{g} : Z \to \widetilde{K}$ such that $\tilde{g}(z_0) = \tilde{z}_0$; see [9, 1.33]. If $Z \subset K$ then by a lifting of Z we understand a lifting of the identity on Z.

The following observation is a simple consequence of Observation 8.9.

Observation 8.10. Suppose Z is path connected, $g : Z \to K$ is continuous, and $\tilde{g}_0$ and $\tilde{g}_1$ are lifting of g. Then there exists exactly one $n \in \mathbb{Z}$ such that $\xi_n \circ \tilde{g}_0 = \tilde{g}_1$.

Observation 8.11. Let $\tilde{g}$ be a lifting of an arc $L \subset K$. Then p restricted to $\tilde{g}(L)$ is a homeomorphism onto L. If $n \neq 0$ is an integer, then $\xi_n \circ \tilde{g}(L) \cap \tilde{g}(L) = \emptyset$.

For each $j \in \mathbb{Z}$ set $\widetilde{K}_j = \left[j/3, (j+1)/3 \right] \times (0, \infty)$.

Observation 8.12. $\{\widetilde{K}_j \mid j \in \mathbb{Z}\}$ has the following properties:

(1) $\widetilde{K} = \bigcup_{j \in \mathbb{Z}} \widetilde{K}_j$,
(2) $\widetilde{K}_{j'} \cap \widetilde{K}_{j''} \neq \emptyset$ if and only if $\left| j' - j'' \right| \leq 1$, and
(3) p restricted to $\widetilde{K}_j$ is a homeomorphism of $\widetilde{K}_j$ onto $K_{j_{(=3)}}$.

Proposition 8.13. *Let k, $\widehat{c'c''}$, $c_0, \ldots, c_{3k}$, α, $u_0, \ldots, u_{3k-1}$ and g be as in Lemma 8.5. Let $\tilde{g} : \widehat{c'c''} \to \widetilde{K}$ be a lifting of g, and let l be an integer such that $\tilde{g}(u_0) \in \widetilde{K}_l$. Then $l_{(=3)} = 0$, and the following conditions are satisfied*

$(1)_j$ $\tilde{g}(u_j) \in \operatorname{int}(\widetilde{K}_{l+j})$ for each $j = 0, \ldots, 3k-1$, and
$(2)_j$ $\tilde{g}(\widehat{u_{j-1}u_j}) \subset (\widetilde{K}_{l+j-1} \cup \widetilde{K}_{l+j}) \setminus (\widetilde{K}_{l+j-2} \cup \widetilde{K}_{l+j+1})$ for each $j = 1, \ldots, 3k-1$.

Additionally, $\tilde{g}(\widehat{c_0 u_0}) \subset (\widetilde{K}_{l-1} \cup \widetilde{K}_l) \setminus (\widetilde{K}_{l-2} \cup \widetilde{K}_{l+1})$ and $\tilde{g}(\widehat{u_{3k-1}c_{3k}}) \subset (\widetilde{K}_{l+3k-1} \cup \widetilde{K}_{l+3k}) \setminus (\widetilde{K}_{l+3k-2} \cup \widetilde{K}_{l+3k+1})$.

Proof. It follows from $8.5(1)_0$ that $l_{(=3)} = 0$ and $\tilde{g}(u_0) \in \operatorname{int}(\widetilde{K}_l)$. So, the condition $(1)_0$ of the proposition is true. We will prove that the implication $(1)_{j-1} \Rightarrow (2)_j$ and $(1)_j$ is true for each $j = 1, \ldots, 3k-1$. For this purpose, suppose $(1)_{j-1}$ is true for some $j = 1, \ldots, 3k-1$. Since $(j-1)_{(-3)} = j_{(+3)}$, by combining $8.5(3)_{j-1}$ with $8.5(2)_j$ we infer that $g(\widehat{u_{j-1}u_j}) \subset (K_{j_{(-3)}} \cup K_{j_{(=3)}}) \setminus K_{j_{(+3)}}$. Since $l_{(=3)} = 0$, p restricted to each of the sets $\widetilde{K}_{l+j-2}$, $\widetilde{K}_{l+j-1}$, $\widetilde{K}_{l+j}$ and $\widetilde{K}_{l+j+1}$ is a homeomorphism of onto $K_{j_{(+3)}}$, $K_{j_{(-3)}}$, $K_{j_{(=3)}}$ and $K_{j_{(+3)}}$, respectively. Consequently, p restricted to $(\widetilde{K}_{l+j-1} \cup \widetilde{K}_{l+j}) \setminus (\widetilde{K}_{l+j-2} \cup \widetilde{K}_{l+j+1})$ is a homeomorphism of onto $(K_{j_{(-3)}} \cup K_{j_{(=3)}}) \setminus K_{j_{(+3)}}$. Hence, $(2)_j$ is true because $\tilde{g}(u_{j-1}) \in \operatorname{int}(\widetilde{K}_{l+j-1})$ by the assumed $(1)_{j-1}$. That implies $\tilde{g}(u_j) \in (\widetilde{K}_{l+j-1} \cup \widetilde{K}_{l+j}) \setminus (\widetilde{K}_{l+j-2} \cup \widetilde{K}_{l+j+1})$. Since $g(u_j) \in \operatorname{int}(K_{j_{(=3)}})$ by $8.5(1)_j$, we infer that $\tilde{g}(u_j) \in \operatorname{int}(\widetilde{K}_{l+j})$. So, the proof of the implication $(1)_{j-1} \Rightarrow (2)_j$ and $(1)_j$ is complete, and conditions $(2)_j$ and $(1)_j$ are true by induction. The proof of the remaining two additional conditions is similar to the above argument and it will be omitted. □

Suppose g is a continuous mapping of an arc L into K, and $\tilde{g}$ is a lifting of g. Set $\mu = \lceil \min(r_1 \circ \tilde{g}(L)) \rceil$ and $\nu = \lfloor \max(r_1 \circ \tilde{g}(L)) \rfloor$. By Observation 8.10, the difference $\nu - \mu$ depends only on g and not on the choice of lifting $\tilde{g}$. So, we may set $\ell(g) = \max(\nu - \mu, 0)$. If $L \subset K$, by $\ell(L)$ we understand $\ell(\mathrm{id}_L)$.

The following proposition is a simple consequence of Observation 8.4.

Proposition 8.14. *Let g be a mapping of an arc L into K. Then L contains a collection $\mathscr{C}$ of $3\ell(L)$ mutually disjoint arcs such that $\operatorname{diam}(g(C)) > 1$ for each $C \in \mathscr{C}$. In particular,* *if $L \subset K$, then L contains a collection of $3\ell(L)$ mutually disjoint arcs each of which has diameter greater than* 1.

Suppose again g is a continuous mapping of an arc $L = \widehat{c'c''}$ into K, $\tilde{g}$ is a lifting of g, and $\mu = \lceil \min(r_1 \circ \tilde{g}(L)) \rceil < \nu = \lfloor \max(r_1 \circ \tilde{g}(L)) \rfloor$. We say that g wraps L counterclockwise around E if there is a collection $C_\mu, C_{\mu+1}, \ldots, C_\nu$ of mutually disjoint subarcs of L such that $c' \in C_\mu$, $c'' \in C_\nu$, and $(r_1 \circ \tilde{g})^{-1}(j) \subset C_j$ for all $j = \mu, \ldots, \nu$. We say that g wraps $\widehat{c'c''}$ clockwise around E if it wraps $\widehat{c''c'}$ counterclockwise. We say that g wraps L around E if it wraps either counterclockwise or clockwise. It follows from Observation 8.10 that the above properties depend only on g and not on the choice of lifting $\tilde{g}$. If $L \subset K$ and the identity wraps L around E (counterclockwise or clockwise), we simply say that L wraps itself around E (counterclockwise or clockwise).

Proposition 8.15. *Suppose g is a continuous mapping of an arc $L = \widehat{c'c''}$ into K wrapping L counterclockwise around E. Let $\tilde{g}$, μ, ν and $C_\mu, C_{\mu+1}, \ldots, C_\nu$ be as in the above definition. For each $j = \mu, \ldots, \nu$, let c'_j and c''_j denote the endpoints of C_j listed in such order that $c'_j < c''_j$ where the inequality reflects the order on L oriented from c' to c''. Then*

$$c' = c'_\mu < c''_\mu < c'_{\mu+1} < c''_{\mu+1} < \cdots < c'_{\nu-1} < c''_{\nu-1} < c'_\nu < c''_\nu = c''$$

Proof. Clearly, $c' = c'_\mu$ and $c''_\nu = c''$. For each $i = \mu, \ldots, \nu - 1$, consider the following statement:

$$(S_i) \qquad c'_\mu < c''_\mu < \cdots < c'_i < c''_i < c'_j \quad \text{for all } j = i+1, \ldots, \nu.$$

Since the arc C_μ contains c' which is the least point in L oriented from c' to c'', we infer that $c' = c'_\mu$, $C_\nu = \widehat{c'c''_\mu}$ and $c''_\mu < c'_j$ for all $j = \mu+1, \ldots, \nu$. So, S_μ is true. On the other hand, since $c''_\nu \in C_\nu$, it follows that $c''_\nu = c''$ and $S_{\nu-1}$ implies the proposition. To complete the proof it is enough to prove the implication $S_i \Rightarrow S_{i+1}$ for all $i = \mu, \ldots, \nu - 2$. For that purpose, suppose that S_i is true, but S_{i+1} is false. Then there is an integer $j = i+2, \ldots, \nu$ such that $c'_i < c''_i < c'_j < c''_j < c'_{i+1}$. Observe that the arc $\widehat{c'_i c''_j}$ contains both C_i and C_j, but it does not intersect C_{i+1}. Consequently, $r_1 \circ \tilde{g}\left(\widehat{c'_i c''_j}\right)$ contains i and j, but it does not contain $i+1$. This contradiction completes the proof of the proposition. □

Corollary 8.16. *Suppose g is a continuous mapping of an arc L into K such that it wraps L around E. Let $\tilde{g}$ be a lifting of g and let $j \in \mathbb{Z}$. Then at most one component of $(r_1 \circ \tilde{g})^{-1}([j, j+1])$ is mapped by $r_1 \circ \tilde{g}$ onto $[j, j+1]$.*

Corollary 8.17. *Suppose g is a continuous mapping of an arc $L = \widehat{c'c''}$ into K such that it wraps L counterclockwise (or clockwise) around E. Let $\widehat{u'u''} \subset \widehat{c'c''}$ be an arc such that $\ell\left(\widehat{u'u''}\right) \geq 1$. Then g restricted to $\widehat{u'u''}$ wraps this arc counterclockwise (or clockwise, respectively) around E.*

The following corollary is a summary of Proposition 8.13.

Corollary 8.18. *Let $k > 2$, $\widehat{c'c''}$, and g be as in Proposition 8.13. Then g wraps $\widehat{c'c''}$ counterclockwise around E, and $k - 2 \leq \ell(g) \leq k$.*

Proposition 8.19. *Let $L = \widehat{c'c''} \subset K$ be an arc and let c be a point in the interior of L such that $\widehat{cc''}$ wraps itself counterclockwise around E. Let $\tilde{g}$ be a lifting of L, let $l_1 = \left\lfloor \max\left(r_1 \circ \tilde{g}\left(\widehat{cc''}\right)\right) \right\rfloor$ and let $l_0 = l_1 - \left(\ell\left(\widehat{cc''}\right) - \ell\left(\widehat{c'c}\right) - 2\right)$. Then*

$$r_1 \circ \tilde{g}\left(\widehat{c'c}\right) \cap [l_0, \infty) = \emptyset.$$

Proof. Since $\left\lceil \min\left(r_1 \circ \tilde{g}\left(\widehat{c'c}\right)\right)\right\rceil - 1 < r_1 \circ \tilde{g}(c)$ and $\widehat{cc''}$ wraps itself counterclockwise around E, it follows that $\left\lceil \min\left(r_1 \circ \tilde{g}\left(\widehat{c'c}\right)\right)\right\rceil - 1 \le \left\lceil \min\left(r_1 \circ \tilde{g}\left(\widehat{cc''}\right)\right)\right\rceil$. Now, we complete the proof of the proposition by the following sequence of equalities and inequalities.

$$\max\left(r_1 \circ \tilde{g}\left(\widehat{c'c}\right)\right) < \left\lfloor \max\left(r_1 \circ \tilde{g}\left(\widehat{c'c}\right)\right)\right\rfloor + 1 = \left\lceil \min\left(r_1 \circ \tilde{g}\left(\widehat{c'c}\right)\right)\right\rceil + \ell\left(\widehat{c'c}\right) + 1 \le$$

$$\le \left\lceil \min\left(r_1 \circ \tilde{g}\left(\widehat{cc''}\right)\right)\right\rceil + \ell\left(\widehat{c'c}\right) + 2 = \left\lfloor \max\left(r_1 \circ \tilde{g}\left(\widehat{cc''}\right)\right)\right\rfloor - \ell\left(\widehat{cc''}\right) + \ell\left(\widehat{c'c}\right) + 2 = l_0$$

□

The next proposition is a dual version of 8.19. We omit its proof since it is essentially the same as that of 8.19.

Proposition 8.20. *Let $L = \widehat{c'c''} \subset K$ be an arc and let c be a point in the interior of L such that $\widehat{c'c}$ wraps itself counterclockwise around E. Let $\tilde{g}$ be a lifting of L, let $l_1' = \left\lceil \min\left(r_1 \circ \tilde{g}\left(\widehat{c'c}\right)\right)\right\rceil$ and let $l_0' = l_1' + \left(\ell\left(\widehat{c'c}\right) - \ell\left(\widehat{cc''}\right) - 2\right)$. Then*

$$r_1 \circ \tilde{g}\left(\widehat{cc''}\right) \cap (-\infty, l_0'] = \emptyset.$$

Lemma 8.21. *Let $L = \widehat{c'c''}$ be an arc contained in $D \setminus E$ such that $L \cap \mathrm{bd}(D) = \{c'\}$. Suppose c, $\tilde{g}$, l_0 and l_1 are as in Proposition 8.19, except that here we require $\ell\left(\widehat{c'c}\right) + 4 \le \ell\left(\widehat{cc''}\right)$. Let v denote the point in the set $L \cap J_0$ which is the closest to e_0. Finally, let l_v be an integer such that $\tilde{g}(v) \in \{l_v\} \times (0, \infty)$. Then l_v is either l_1 or $l_1 - 1$.*

Proof. Notice that $r_2 \circ \tilde{g}(c') = 1$ since $p^{-1}(D) = \mathbb{R} \times (0, 1]$ and $c' \in \mathrm{bd}(D)$. Denote $r_1 \circ \tilde{g}(c')$ by u. So, $g(c') = (u, 1)$. By Proposition 8.19, $u < l_0 = l_1 - \left(\ell\left(\widehat{cc''}\right) - \ell\left(\widehat{c'c}\right) - 2\right) \le l_1 - 2$.

Let V denote the component of $J_0 \setminus \{v\}$ whose closure contains e_0. Clearly, $V \cap L = \emptyset$. Set $\widetilde{V} = p^{-1}(V) \cap (\{l_v\} \times (0, 1])$. Observe that p restricted to $\widetilde{V}$ is a homeomorphism onto V since $\omega(\{e_0\} \times (0, \infty)) = J_0$. So, $\widetilde{V}$ is an open arc contained in $\{l_v\} \times (0, 1] \subset \{l_v\} \times [0, 1]$ such that one of its ends is $\tilde{g}(v)$ and the other is $(l_v, 0)$. It follows that $\widetilde{W} = \tilde{g}\left(\widehat{c'v}\right) \cup \widetilde{V}$ is an arc (closed in one side and open on the other) that separates $p^{-1}(D)$ into two components. We denote them by C_- and C_+ such that $(-\infty, u) \times \{1\} \subset C_-$ and $(u, \infty) \times \{1\} \subset C_+$. By Observation 8.11, the arc $\xi_{-1} \circ \tilde{g}(L)$ does not intersect $\widetilde{W}$. Since $\xi_{-1} \circ \tilde{g}(c') = (u - 1, 1)$, the arc $\xi_{-1} \circ \tilde{g}(L)$ is contained in C_-. Since $l_1 \in r_1 \circ \tilde{g}(L)$, it follows that $l_1 - 1 \in r_1 \circ \xi_{-1} \circ \tilde{g}(L)$. Consequently, $l_1 - 1 \in r_1(C_-)$. Thus, there exists a point z in the interior of the arc $\widehat{c'v}$ such that $r_1 \circ \tilde{g}(z) > l_1 - 1$. Since $u < l_0 \le l_1 - 2$, there is a point w in the interior of $\widehat{c'z}$ such that $r_1 \circ \tilde{g}(w) = l_1 - 2$. It follows from Proposition 8.19 that $\widehat{wv} \subset \widehat{cc''}$ and, consequently, $\widehat{wv}$ wraps itself counterclockwise around E. Since $r_1 \circ \tilde{g}(w) = l_1 - 2$, $r_1 \circ \tilde{g}(z) > l_1 - 1$, $z \in \widehat{wv}$ and $r_1 \circ \tilde{g}(v) = l_v$, it follows from Corollary 8.16 that $l_v \ge l_1 - 1$. Hence, the lemma is true. □

Proposition 8.22. *Suppose $L \subset D \setminus E$ is an arc with endpoints c' and c'' such that $L \cap \mathrm{bd}(D) = \{c', c''\}$. Let $\tilde{g} : L \to \widetilde{K}$ be a lifting of L. Then $\left|r_1 \circ \tilde{g}(c') - r_1 \circ \tilde{g}(c'')\right| < 1$.*

Proof. Let C_1 and C_2 denote the two subarcs of $\mathrm{bd}(D)$ with endpoints c' and c''. Observe that exactly one of the simple closed curves $L \cup C_1$ and $L \cup C_2$, say $L \cup C_1$, bounds a disk in the plane that does not intersect E. Let $\tilde{g}_1$ be the lifting of C_1 to $\widetilde{K}$ such that $\tilde{g}_1(c') = \tilde{g}(c')$. Then $\tilde{g}_1(c'') = \tilde{g}(c'')$ by [22, Th. 54.3]. Now, the proposition follows from Observations 8.7 and 8.8. □

Proposition 8.23. *Let $L \subset D \setminus E$ be an arc with endpoints c' and c'' such that $L \cap \mathrm{bd}(D) = \{c', c''\}$. Suppose $u, v \in L$ are such that $c' \le u < v \le c''$ and $\widehat{uv}$ wraps itself around E. Then*

$$\ell\left(\widehat{uv}\right) \le \ell\left(\widehat{c'u}\right) + \ell\left(\widehat{vc''}\right) + 4.$$

Proof. We may assume without loss of generality that $\widehat{uv}$ wraps itself counterclockwise around E. (In the other case we could just reverse the orientation on L.) Let $\tilde{g} : L \to \widetilde{K}$ be a lifting. Setting $l_1 = \left\lfloor \max\left(r_1 \circ \tilde{g}\left(\widehat{uv}\right)\right)\right\rfloor$ and using Proposition 8.19 with $c' = c'$, $c = u$ and $c'' = v$ we infer that

$$r_1 \circ \tilde{g}\left(c'\right) < l_1 - \ell\left(\widehat{uv}\right) + \ell\left(\widehat{c'u}\right) + 2$$

Setting $l_1' = \left\lceil \min\left(r_1 \circ \tilde{g}\left(\widehat{uv}\right)\right)\right\rceil$ and using Proposition 8.20 with $c' = u$, $c = v$ and $c'' = c''$ we infer that

$$l_1' + \ell\left(\widehat{uv}\right) - \ell\left(\widehat{vc''}\right) - 2 < r_1 \circ \tilde{g}\left(c''\right)$$

By adding the above inequalities, and then moving l_1, l_1' and $\ell\left(\widehat{uv}\right)$ to the left side of the resulting inequality, and all the remaining terms to the right side we infer that

$$l_1' - l_1 + 2\ell\left(\widehat{uv}\right) < \ell\left(\widehat{vc''}\right) + \ell\left(\widehat{c'u}\right) + 4 + r_1 \circ \tilde{g}\left(c''\right) - r_1 \circ \tilde{g}\left(c'\right)$$

Since $l_1' - l_1 = -\ell\left(\widehat{uv}\right)$, the left side of the last inequality equals $\ell\left(\widehat{uv}\right)$. Thus

$$(\star) \qquad \ell\left(\widehat{uv}\right) < \ell\left(\widehat{vc''}\right) + \ell\left(\widehat{c'u}\right) + 4 + r_1 \circ \tilde{g}\left(c''\right) - r_1 \circ \tilde{g}\left(c'\right)$$

Observe that $\ell\left(\widehat{uv}\right)$ and $\ell\left(\widehat{vc''}\right) + \ell\left(\widehat{c'u}\right) + 4$ are integers. Since $r_1 \circ \tilde{g}\left(c''\right) - r_1 \circ \tilde{g}\left(c'\right) < 1$ (by Proposition 8.22) we may remove the difference $r_1 \circ \tilde{g}\left(c''\right) - r_1 \circ \tilde{g}\left(c'\right)$ from the inequality $(\star)$ while replacing "$<$" by "$\le$". So, the proposition is true. □

Lemma 8.24. *Suppose N is a positive integer. Let $\widehat{c'c}$ be an arc contained in $D \setminus E$ such that $\widehat{c'c} \cap \mathrm{bd}(D) = \{c'\}$. Suppose u is a point in the interior of L such that $\ell\left(\widehat{c'u}\right) \le N$, $\ell\left(\widehat{uc}\right) \ge 2N + 6$ and $\widehat{uc}$ wraps itself counterclockwise around E. Finally, suppose $Z \subset D \setminus (L \cup E)$ is a set with the property that for each $z \in Z$ there is an arc $L_z \subset Z$ such that $z \in L_z$, $L_z \cap \mathrm{bd}(D) \neq \emptyset$ and $\ell(L_z) \le N$. Then Z does not separate D between L and E.*

Proof. Let v denote the point in the set $\widehat{c'c} \cap J_0$ which is the closest to e_0. It follows from Proposition 8.19 and Lemma 8.21 that u is in the interior of $\widehat{c'v}$ and $\ell\left(\widehat{uv}\right)$ is either $\ell\left(\widehat{uc}\right)$ or $\ell\left(\widehat{uc}\right) - 1$. Consequently,

$$(\ge) \qquad \ell\left(\widehat{uv}\right) \ge 2N + 5$$

Let V denote the component of $J_0 \setminus \{v\}$ whose closure contains e_0. Clearly, $V \cap L = \emptyset$ and $\widehat{uv} \cup V \cup \{e_0\}$ is an arc. If $Z \cap V = \emptyset$ then the lemma is true. So, we may assume that $Z \cap V$ contains a point z. Then there is an arc $L_z \subset Z$ such that $z \in L_z$, $L_z \cap \mathrm{bd}(D) \neq \emptyset$ and $\ell(L_z) \le N$. Let $\widehat{z_0 c''}$ be a subarc of L_z minimal with respect to the property: $z_0 \in V$ and $c'' \in \mathrm{bd}(D)$. Let $\widehat{vz_0}$ denote the subarc of J_0 with endpoints v and z_0. Consider the arc $\widehat{vc''} = \widehat{vz_0} \cup \widehat{z_0 c''}$. Since $\widehat{vz_0} \subset J_0$, it follows that $\ell\left(\widehat{vc''}\right) = \ell\left(\widehat{z_0 c''}\right) \le \ell(L_z) \le N$. Now, we consider the arc $L = \widehat{c'c''} = \widehat{c'u} \cup \widehat{uv} \cup \widehat{vc''}$ and apply Proposition 8.23 to get the result that

$$(\le) \qquad \ell\left(\widehat{uv}\right) \le \ell\left(\widehat{c'u}\right) + \ell\left(\widehat{vc''}\right) + 4 \le 2N + 4$$

It follows that $Z \cap V = \emptyset$ since the the inequalities $(\ge)$ and $(\ge)$ contradict each other. Hence, the lemma is true. □

9. PART 2 OF THE DEFINITION OF X

In this section we specify the winding numbers in the definition of X and therefore fully define continuum X.

Proposition 9.1 (see [21, Proposition 2.1])**.** *Suppose L is an arc and $\epsilon > 0$. Then there is a positive integer $N(L,\epsilon)$ such that, for each collection $\mathscr{C}$ of $N(L,\epsilon)$ subarcs of L whose interiors are mutually disjoint, at least one element of $\mathscr{C}$ has diameter less than ϵ.*

In the preliminary definition of X given in Subsection 4.3, we used a generic sequence of positive integers $\Sigma = (k_0, k_1, \dots)$ without any other restrictions. This was enough to prove the basic properties of X. However, we need to impose some conditions on Σ to be able to prove that X admits a simple dense canal for every embedding into $\mathbb{R}^2$. We will define the terms of Σ one by one starting from k_0. For each nonnegative integer n, we will define k_n basing on properties of some arcs contained in $X \subset \prod_{j=0}^{n} X_j$ in such a way that their complete definitions do not depend on k_l for any $l \geq n$. However, the arcs used to define k_1 may depend on k_0, the arcs used to define k_2 may depend on k_0 and k_1, and so on. Having in mind the inductive character of the the construction, we will define k_n in two cases depending whether n is odd or even.

Let $\nu \in \{0,1,2,3\}$ and let n be a nonnegative integer. Recall that $\widehat{a^\nu a_n^\nu} \subset F_n$ the arc $\left[\widehat{a^\nu a_n^\nu}\right]$ does not depend on k_l for any $l \geq n$; see Observation 6.7. Set

$$N_n = \max\left\{N\left(\left[\widehat{a^\nu a_n^\nu}\right], 2^{-n}\right) \mid \nu \in \{0,1,2,3\}\right\}$$

where $N(\cdot,\cdot)$ is the number defined in Proposition 9.1. We will use N_n in the definition of k_n in both cases of odd and even n. Set

$$k_n = 2\max(N_n, n), \quad \text{if } n \text{ is odd.}$$

Now, suppose n is even and set $i = n/2$. Recall that $f_n : X_{n+1} \to X_n$ where $X_n = \widetilde{A}_i$, $X_{n+1} = \widetilde{A}_{i+1}$ and $f_n = \varphi_{n,k_n}$. Also recall that τ is an involution of $\{0,1,2,3\}$ such that $\tau(0) = 1$, $\tau(1) = 0$, $\tau(2) = 3$ and $\tau(3) = 2$. It follows from Proposition 6.8 that $\widehat{s_{i+1}^\nu b_{i+1}^\nu} \subset F_{n+1}$. Recall that $\underline{b}_{i+1}^\nu$ is a point in the interior of $\widehat{s_{i+1}^\nu b_{i+1}^\nu}$, and, if $\nu \in \{1,3\}$ then $\underline{s}_{i+1}^\nu$ is a point in the interior of $\widehat{s_{i+1}^\nu \underline{b}_{i+1}^\nu}$. Also, recall that $\mathscr{L}_{i+1}$ is a collection of arcs defined before Corollary 6.9. By the corollary, each of the four arcs from $\mathscr{L}_{i+1}$ does not depend on k_l for any $l \geq n$. Let $N_n^e = \max\left\{N(L, 2^{-n}) \mid L \in \mathscr{L}_{i+1}\right\}$. Set

$$k_n = 2\max\left(N_n, N_n^e, n\right), \quad \text{if } n \text{ is even.}$$

This completes the construction of the sequence $\Sigma = (k_0, k_1, \dots)$. Therefore, X is now fully defined.

10. DEFINITION OF THE SIMPLE DENSE CANAL

We first recall parts of the Prime End Theory needed for the setup. We refer to the paper by Brechner [5] for more detailed description. Let $S^2 \subset \mathbb{R}^3$ denote a unit sphere. We denote by $B^1 \subset S^2$ the unit disk.

Definition 10.1. Let $U \subset S^2$ be a simply connected open set with a nondegenerate boundary. A *crosscut* Q of U is an open arc in U such that $\mathrm{cl}(Q)$ intersects $\mathrm{bd}(U)$ in exactly two endpoints of $cl(Q)$. A *C-map* $\psi : U \to \mathrm{int}\left(B^1\right)$ is a homeomorphism such that:

(1) $\psi(U)$ is a crosscut of $\mathrm{int}\left(B^1\right)$.

(2) $\{\psi(\mathrm{bd}(Q)) \mid Q$ is a crosscut of $U\}$ is dense in $\mathrm{bd}(B^1)$.

A sequence of crosscuts $Q_1, Q_2, \dots$ of U is a *chain of crosscuts* if and only if all three of the following conditions hold:

(1') $\mathrm{cl}(Q_1), \mathrm{cl}(Q_2), \dots$ are pairwise disjoint.
(2') Q_n separates Q_{n-1} from Q_{n+1} in U.
(3') $\mathrm{diam}(Q_i) \to 0$ and $\lim(Q_i)$ is a point as $i \to \infty$.

Let U_n be a simply connected open set that contains Q_{n+1}. We refer to U_n by *inner domains*. Thus $U_1 \subset U_2 \subset U_3 \subset \dots$. Let $Q_1, Q_2, \dots$ and $R_1, R_2, \dots$ be chains of crosscuts of U and $U_1, U_2, \dots$ and $V_1, V_2, \dots$ their respective corresponding inner domains. Then $Q_1, Q_2, \dots$ and $R_1, R_2, \dots$ are equivalent chains of crosscuts of U if and only if for every positive integer i there exist a positive integer j so that $V_j \subset U_i$ and $U_j \subset V_i$. A *prime end $\mathcal{E}$ of $S^2 \setminus U$* is an equivalence class of chains of crosscuts of U.

Theorem 10.2. *Let $U \subset S^2$ be a simply connected open set with a nondegenerate boundary. Then there exists a C-map $\psi : U \to int(B^1)$. If $\mathcal{E}$ is a prime end determined by a chain of crosscuts $Q_1, Q_2, \dots$ with corresponding inner domains $U_1, U_2, \dots$ and $p \in bd(B^1)$ is the point corresponding to $\mathcal{E}$, then a sequence of points $z_1, z_2, \dots$ in U has the property that $z_i \in U_i$ for every positive integer i if and only if $\psi(z_i)$ converges to p as $i \to \infty$.*

Definition 10.3. Let $U \subset \mathbb{R}^2$ be a simply connected open set with a nondegenerate boundary and $R \subset U$ be a ray. Let $r \in R$ be a point. A crosscut Q is called a *transverse crosscut to R at r*, if there exists a topological disk $D \subset U$ such that $r \in \mathrm{int}(D)$ and $(R \cap D) \setminus \{r\}$ consists of two components, each of which is contained in exactly one simply connected component of $D \setminus Q$. A continuum $K \subset \mathbb{R}^2$ has a *simple dense canal*, if there exists a ray $R \subset \mathbb{R}^2 \setminus K$ such that the following three conditions hold:

(1) $cl(R) \setminus R = K$.
(2) for every point $r \in R$ there exists a sequence of transverse crosscuts to R at r.
(3) diameter of transverse crosscuts from (2) converges to 0.

11. Any embedding of X into the plane has a dense simple canal

Let h be an arbitrary embedding of X into the plane. Since $[T^*]$ and the four spurs S^ν are mutually disjoint tree-like continua, there are five mutually disjoint closed topological disks D^*, D^0, D^1, D^2 and D^3 contained in the plane such that $h([T^*]) \subset \mathrm{int}(D^*)$ and $h([S^\nu]) \subset \mathrm{int}(D^\nu)$ for all $\nu \in \{0,1,2,3\}$. Using Remark (i) after the proof of [13, Theorem 6,§61,IV] we can find a homeomorphism g of the plane onto itself such that

- g maps each of the disks D^*, D^0, D^1, D^2 and D^3 onto a circular disk with radius 2,
- $g \circ h([T^*])$ is the standard unit triod with ordered set of endpoints $g \circ h([t_0^*])$, $g \circ h([t_1^*])$ and $g \circ h([t_2^*])$ and its center is at the center of $g(D^*)$, and
- for each $\nu \in \{0,1,2,3\}$, $g \circ h([Y^\nu])$ is the standard unit triod with ordered set of endpoints $g \circ h([y_0^\nu])$, $g \circ h([y_1^\nu])$ and $g \circ h([y_2^\nu])$ and its center at the center of $g(D^\nu)$.

Since $h(X)$ has a dense simple canal if and only if $g \circ h(X)$ has a dense simple canal, we may assume that

(1) D^*, D^0, D^1, D^2 and D^3 are mutually disjoint closed circular disks, each with radius 2 and such that $h([T^*]) \subset \mathrm{int}(D^*)$ and $h([S^\nu]) \subset \mathrm{int}(D^\nu)$ for all $\nu \in \{0,1,2,3\}$.
(2) $h([T^*])$ is the standard unit triod with ordered set of endpoints $h([t_0^*])$, $h([t_1^*])$ and $h([t_2^*])$ and its center is at the center of D^*, and

(3) for each $\nu \in \{0,1,2,3\}$, $h([Y^\nu])$ is the standard unit triod with ordered set of endpoints $h([y_0^\nu])$, $h([y_1^\nu])$ and $h([y_2^\nu])$ and its center at the center of D^ν.

Let D_∞ be a big circular disk in the plane such that D^*, D^0, D^1, D^2 and D^3 are contained in its interior. Let a_∞ be a point in the boundary of D_∞. Recall that, for each $\nu \in \{0,1,2,3\}$, $h(a^\nu)$ is accessible from the complement of $h(X)$; see Proposition 7.9. It follows that, for each $\nu \in \{0,1,2,3\}$, there is $\widehat{a_\infty a^\nu} \subset D_\infty$ such that $\widehat{a_\infty a^\nu} \cap h(X) = \{a^\nu\}$, and $\widehat{a_\infty a^\nu} \cap \widehat{a_\infty a^\mu} = \{a_\infty\}$ for $\mu \in \{0,1,2,3\} \setminus \{\nu\}$.

Remark 11.1. What remains to be done to complete the proof is an explicit construction of a simple dense canal in an embedding $h(X)$. In the construction we will use Section 8 as the main tool and heavily rely on the specific inductive choice of the sequence of wrapping numbers Σ. Since the construction is not complete yet, it is omitted in the current version of the file.

REFERENCES

1. Anderson, R. D. Choquet, G. *A plane continuum no two of whose nondegenerate subcontinua are homeomorphic: an application of inverse limits*, Proc. Am. Math. Soc. **10**, (1959), 347–353.
2. Bell, H. *On fixed point properties of plane continua*, Trans. Amer. Math. Soc. **128** (1967), 539–548.
3. Bellamy, D. P. *A tree-like continuum without the fixed-point property*, Houston J. Math. **6** (1981), no. 1, 1–13.
4. Bing R. H. *The elusive fixed point property*, Amer. Math. Monthly **76** (1969), 119–132.
5. Brechner, B. *On stable homeomorphisms and imbeddings of the pseudo arc*, Illinois J. Math. **22**, no. 4 (1978), 630–661.
6. Brechner, B. Mayer, J. C. *The prime end structure of indecomposable continua and the fixed point property*, General topology and modern analysis (Proc. Conf., Univ. California, Riverside, Calif., 1980), 151–168, Academic Press, New York-London, 1981.
7. Brown M. *Some applications of an approximation theorem for inverse limits*, Proc. Am. Math. Soc. **11** (1960), no. 3, 478–483.
8. Fearnley, L. Wright, D. G. *Geometric realization of Bellamy continuum* Bull. London Math. Soc. **25** (1993), no. 2, 177–183.
9. Hatcher, A. *Algebraic Topology,* Cambridge University Press, 2002.
10. Hernández-Gutiérrez R. Hoehn, L. *A fixed-point-free map of a tree-like continuum induced by bounded valence maps on trees*, arXiv:1608.08094.
11. Hagopian C. L. *An update on the elusive fixed-point property*, Open problems in topology II (Elliott Pearl, ed.), Elsevier B. V. Amsterdam, 2007, pp. 263–277.
12. Iliadis S. *Positions of continua on the plane and fixed points,* Vestnik Moskov. Univ. Ser. I Math. Mekh. (1970), 66–70.
13. K. Kuratowski, *Topology, volume II* Academic Press and PWN - Polish Scientific Publishers, 1968.
14. Lewis, W. *Continuum Theory Problems*, Topol. Proc. **8** (1983), 361–394.
15. Mauldin, R. D. *The Scottish Book,* Birkhäuser, Boston, 1981.
16. Minc P. *A hereditarily indecomposable tree-like continuum without the fixed point property*, Trans. Amer. Math. Soc. **352** (2000), no. 2, 643–654.
17. Minc P. *A periodic point free homeomorphism of a tree-like continuum*, Trans. Amer. Math. Soc. **348** (1996), no. 4, 1487–1519.
18. Minc P. *A self-map of a tree-like continuum with no invariant indecomposable subcontinuum*, Topology Appl. **98** (1999), no. 1-3, 235–240, II Iberoamerican Conference on Topology and its Applications (Morelia, 1997).
19. Minc P. *A tree-like continuum admitting fixed point free maps with arbitrarily small trajectories*, Topology Appl. **46**, (1992), no. 2, 99–106.
20. Minc P. *A weakly chainable tree-like continuum without the fixed point property*, Trans. Amer. Math. Soc. **351** (1999), no. 3, 1109–1121.
21. Minc, P. Sturm, F. *Homeomorphism killing rays*, Houston J. Math. **41** (2015), 1341–1350.
22. Munkres J. R. *Topology (Second Edition),* Prentice Hall, Upper Saddle River, NJ, 2000.
23. Nadler, S. B. Jr., *Continuum Theory: An Introduction*, Monographs and Textbooks in Pure and Applied Mathematics, 158. Marcel Dekker, Inc., New York, 1992.

24. Oversteegen L. G. Rogers, J. T. Jr., *An inverse limit description of an atriodic tree-like continuum and an induced map without a fixed point*, Houston J. Math. **6**, (1980), no. 4, 549–564.
25. Oversteegen L. G. Rogers, J. T. Jr., *Fixed-point-free maps on tree-like continua*, Topology Appl. **13**, (1982), no. 1, 85âĂŞ95.
26. Sieklucki, K. *On a class of plane acyclic continua with the fixed point property,* Fund. Math. **63**, no. 3, (1968) 257–278.

(J. Činč) FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, A-1090 VIENNA, AUSTRIA
E-mail address: `jernej.cinc@univie.ac.at`