Marshall Plan Scholarship 2009
Final Report

to the

Marshall Plan Foundation
Ungargasse 37, 1030 Wien

submitted by

Gwendolin Wilke
Institute for Geoinformation and Cartography
Vienna University of Technology
Gußhausstrasse 27 - 29 / E 127-1
1040 Vienna, Austria

Phone: +43 1 58801-12713
Fax: +43 1 58801-12799
Email: wilke@geoinfo.tuwien.ac.at

Contents

1 General Information

Scholarship holder:	Gwendolin Wilke Währingerstr. 51/11 1090 Vienna, Austria Phone: +43 1 58801-12713 Mobile: +43 681-1027 3163 Fax: +43 1 58801-12799 Email: wilke@geoinfo.tuwien.ac.at
Period of time:	02/15/2009 - 07/15/2009
Field of study:	Doctoral program in geoinformation
Research topic:	Spray Can Geometry - Towards an Axiomatic Approach to Uncertainty Modeling for Vector Based Geographic Information Systems
Home institution:	Institute for Geoinformation and Cartography Vienna University of Technology Gußhausstrasse 27 - 29 / E 127-1 1040 Vienna, Austria Web: http://www.geoinfo.tuwien.ac.at/
Advisor at home institution:	Prof. Andrew U. Frank Phone: +43 1 58801-12713 Fax: +43 1 58801-12799 Email: frank@geoinfo.tuwien.ac.at
Host Institution:	Berkeley Initiative in Soft Computing (BISC), Department of Electrical Engeneering and Computer Science, Computer Science Division (CS), University of California, Berkeley 387 Soda Hall Berkeley, CA 94720-1776 Phone: (510) 642-1042 FAX: 510-642-5775 Web (CS): http://www.cs.berkeley.edu/ Web (BISC): http://www.cs.berkeley.edu/~zadeh/
Advisor at host institution:	Prof. Lotfi A. Zadeh Phone: +1 510 642-495 Fax: +1 510 642-1712 Email: zadeh@eecs.berkeley.edu

2 Introduction

This is the final report of a five months research visit at the Berkeley Initiative in Soft Computing (BISC), Department of Electrical Engineering and Computer Science, University of California, Berkeley. The research visit was funded by the Marshall Plan Scholarship of the Austrian Marshall Plan Foundation for spring 2009. The results and findings accomplished during the research visit contribute to the scholarship holder's dissertation program in geoinformation, cunducted at the Institute of Geoinformation and Cartography at the Vienna University of Technology.

The remainder of the report is organized as follows: Chapter 3 restates the scholarship holder's PhD topic and research hypothesis and relates it to the research conducted at BISC. Chapter 4 lists the objectives of the research visit. Chapter 5 lists and evaluates the courses and events that have been attended by the scholarship holder during her stay at UC Berkeley. Chapter 6 gives an overview of the research process during the scholarship period and discusses the results briefly. After a short summary in chapter 7, chapter 8 concludes the report with acknowledgements. The appendix contains a presentation and two publications that have been compiled during the research visit.

3 PhD Topic and Research Hypothesis

Spatial analysis is one of the key tools of geographic information systems (GIS). In vector based GIS spatial analysis is based on the algebraic operations of Cartesian geometry. Cartesian geometry is an idealization that represents real world objects by polygons, which comprise of infinitely small points and and infinitely thin lines. This is in contrast to the fact that the location of borders and vertices of geographic objects is inherently uncertain [13], [10]. Consequently, their representation in a GIS is only an approximation.

In general current GIS software does not store information on the approximation error. In the cases where this information is stored, the error is usually not propagated through the geometric construction processes which is the basis for spatial analysis. Very few software tools exists, that propagates statistical error through geometric constructions, e.g. [5], and no tool is able to propagate possibilistic or veristic positional information.

The scholarship holder's PhD research is concerned with laying the foundations for a tool that is capable of propagating arbitrary modalities of uncertainty through geometric constructions in a vector based GIS: Uncertainty degrees are introduced at the lowest level of geometric reasoning, at the level of the axioms of Euclidean geometry.

The presented research proposes a method to propagate arbitrary modalities of error through the process of spatial analysis by assigning a degree of uncertainty to the geometric axioms themselves. Each step of a geometric construction process used during a spatial analysis task can be seen as an inference step from a set of axioms. Fuzzy aproximate reasoning with linguistic variables provides a system of book-keeping and combining the uncertainty degrees of the respective axiom set in every step.

PhD topic:

Spray Can Geometry - An Axiomatic Approach to Geometric Uncertainty Modeling for Vector Based Geographic Information Systems

Research Hypothesis:

An axiomatic model of geometry can be established that incorporates positional uncertainty of objects and operations. The model reflects the extended character of geographic features and the uncertainty in their representation in geographic information systems. A degree of uncertainty is assigned to the axioms using linguistic variables.

The theory of linguistic variables and fuzzy if-then-rules was introduced by Lotfi A. Zadeh in [22], [23], [24], [25] as a means of reasoning with imprecise knowledge. It is part of the framework of the so-called *fuzzy logic in the broader sense*, or *approximate reasoning*. In the proposed research, approximate reasoning techniques are used to formalize approximate inference at the axiomatic level of Euclidean geometry. Prof. Zadeh is head of the BISC research group.

4 Objectives of the Research Visit at BISC

The objectives of the research visit at the BISC research group were

- to deepen knowledge and experience in the field of fuzzy logic and approximate reasoning theory and application,
- to establish closer contact to and collaboration with the BISC research group,
- to apply the methodology of approximate reasoning to the problem domain of the scholarship holder's PhD topic in order to test the research hypothesis, and

- to communicate the methodology to colleagues at the home institution and collaborate in applying it to the department's core research topics.

5 Courses and Events

5.1 Courses

In contrast to the initial scholarship application, the scholarship period did not cover the entire spring semester at UC Berkeley: Due to an administrative delay in the application process, the start of the scholarship period had been postponed for two months, from February 15th, 2009, to April 15th, 2009. For this reason, lectures at undergraduate level, instead of graduate level courses, have been chosen for auditing.

Having the status of a *visiting scholar* at UC berkeley, the scholarship holder was not eligible of enrolling in courses as a regular student, but to *audit* certain courses in consultation with the responsible teachers. Three undergraduate lectures and one graduate level group seminar have been audited. In the following, these courses are listed, together with a short evaluation of each course.

ClassicalGeometries

Lecturer: Vera Serganova
Course Number: MATHEMATICS 130 P 001 LEC
Description: A critical examination of Euclid's Elements; ruler and compass constructions; connections with Galois theory; Hilbert's axioms for geometry, theory of areas, introduction of coordinates, non-Euclidean geometry, regular solids, projective geometry.
Evaluation: The content of this lecture related closely to the scholarship holder's PhD topic of axiomatic uncertainty modelling of geometric reasoning in geographic information systems. The course put a great stress on the axiomatic setup of geometric theories and the resulting hirarchical structure. Interdependencies were discussed in detail. The understanding of Euclidean and non-Euclidean geometries was deepened considerably.

Introduction to Artificial Intelligence

Lecturer: John DeNero
Course Number: COMPUTER SCIENCE 188 P 001 LEC
Description: Basic ideas and techniques underlying the design of intelligent computer systems. Topics include heuristic search, problem solving, game playing, knowledge representation, logical inference, plan-

ning, reasoning under uncertainty, expert systems, learning, perception, language understanding.
Evaluation: The course offered a broad overview of soft computing techniques in artificial intelligence. In addition to the lecture appointments, John DeNero offered tutorials, where the single topics where discussed in more detail. DeNero put an emphasis on baysian methods and touched upon the subject of fuzzy sets theory and fuzzy reasoning only shortly. Baysian methods, and Baysian graphical networks in particular, offer an alternative to fuzzy approximate reasoning for many applications. As a result of the tutorial discussions, the usefulness of fuzzy methods for the problem domain of axiomatic modelling of geometric reasoning for geographic information systems was consolidated.

Probability and Random Processes

Lecturer: Anantharam
Course Number: ELECTRICAL ENGINEERING 126 P 001 LEC
Description: Some knowledge of real analysis and metric spaces, including compactness, Riemann integral. Knowledge of Lebesgue integral and/or elementary probability is helpful, but not essential, given otherwise strong mathematical background. Measure theory concepts needed for probability. Expectation, distributions. Laws of large numbers and central limit theorems for independent random variables. Characteristic function methods. Conditional expectations; martingales and theory convergence. Markov chains. Stationary processes.
Evaluation: The course offered a basic introduction to probability theory and it's applications for electrical engineering problems. The main benefit for the scholarship holder was to get introduced to the theory of random processes and to revise the subject of probability theory with an emphasis on applications.

Group Studies Seminar

Lecturer: Lotfi A. Zadeh
Course Number: COMPUTER SCIENCE 298 P 011 LEC
Description: Advanced study in various subjects through seminars on topics to be selected each year, informal group studies of special problems, group participation in comprehensive design problems, or group research on complete problems for analysis and experimentation.

04/16/2009: Dr. Dilek Hakkani-Tur, International Computer Science Institute (ICSI), *Syntactic and Semantic Graphs for Information Distillation.*

04/28/2009: Dr. Julia Taylor, Purdue University and RiverGlass Inc., *Recognizing Text Based Humor Through The Use Of Ontologies*

06/11/2009: Professor Lotfi A. Zadeh, EECS UC Berkeley, *Toward Extended Fuzzy Logic—A First Step*

06/12/2009: Professor Rafik Aliev, Head of the Department of Computer-Aided Control Systems, Azerbaijan State Oil Academy, Baku, *Decision Analysis with imprecise probabilities*

06/18/2009: Dr. Vilem Novak, Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, *Analysis and Forecasting Time Series using Soft Computing Techniques*

Evaluation: The one hour presentations were followed by a one hour discussion of the topic and about related literature. Prof. Zadeh's and Dr. Novak's talks were of particular interest for the research topic of the scholarship holder.

5.2 Round Table Discussions

The BISC group organizes weekly *Round Table Discussions*. The discussions aim at fostering interdisciplinary collaborations between researchers of different fields who are using soft computing teechniques in their research. Each week, a researcher is invited to be discussion leader. As an impulse, the discussion leader gives a brief and informal outline of the key questions in his or her research field and the role of soft computing techiques in this context.

In the following the Round Table Discussion appointments are listed that have been attended by the scholarship holder:

03/03/2009: Prof. Fumio Mizoguchi, Science University of Tokyo. user preference learning based on fuzz reasoning

03/11/2009: Prof. Martin Wainwright, Department of Statistics, UC Berkeley. Topic: statistical machine learning

03/18/2009: Dr. Sue Liu, Center for Study of Language and Information (CSLI), Stanford University. Topic: text summarization

04/01/2009: Prof. Murat Arcak, College of Engineering, UC Berkeley. Topic: control in systems of different scale

05/13/2009: Gerald Friedland, International Computer Science Institute (ICSI)

05/15/2009: Prof. Sanjoy Mitter, Department of Electrical Engineering, MIT. Topic: feedback and control

05/19/2009: Professor Claire Tomlin, College of Engineering, UC Berkeley. Topic: modeling biological cell networks

05/27/2009: Dr. Monika Ray, UC Davis School of Medicine

06/05/2009: Gwen Wilke, Institute for Geoinformation and Cartography, TU Vienna Topic: positional uncertainty in geographic information systems (slides see Appendix)

06/18/2009: Dr. Irina Perfilieva, Institute for Research and Applications of Fuzzy Modeling, University of Ostrava. Topic: fuzzy transform

Evaluation: The BISC Round Table Discussions provided an excellent opportunity to get insight in related fields of research and make contact with representatives of other institutions concerned with uncertainty modelling. The discussions with Dr. Irina Perfilieva and Dr. Vilem Novak from the Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, were of particular interest for the scholarship holder. A collaboration and/or a research visit at the University of Ostrava in the following academic year has been considered.

5.3 Miscellaneous

Regular meetings with the advisor: The scholarship holder held regular meetings with Prof. Zadeh for a critical review and discussion of the progress concerning the research objective. The meetings took place about every second week. Prof. Zadeh gave very helpful suggestions for the general direction of the research as well as detailed advice. He expressed his satisfaction with the work done so far.

04/18/2009: Math Cal Day: Prof. Robin Hartshorne on *Impossibility in Mathematics*

05/19/2009: Visited the UC Berkeley Geospatial Innovation Facility (GIF)

05/13/2009: EECS Joint Colloquium Distinguished Lecture Series: Prof. Daphne Koller on *Probabilistic Models for Holistic Scene Understanding*

6 Research Process and Outcome

The *research* objective for the research visit at the BISC group was to deepen knowledge and experience in the theory and applications of fuzzy logic and approximate reasoning, and to apply it to the field of geographic information science. In particular the problem domain of geometric reasoning under uncertainty in vector based geographic informations systems has been adressed.

6.1 Research Process

Starting point:

In the work prior to the research visit at the BISC group existing geometric models of positional uncertainty have been systematically tested for compliance with the axioms of projective and Euclidean geometry, respectively. All models violated the respective algebras and it was not possible for the author to modify the models in way that eliminates the flaws and maintain the intended deductive appartus of geometry at the same time.
It was concluded that an exact deduction apparatus does not provide the necessary explication power to describe perception-based reasoning with inexact geometric objects. The framework of fuzzy approximate reasoning has been proposed as a possibly appropriate tool for formulating inexact reasoning procedures in the context of geometric uncertainty modeling for GIS.

Methodology:

As a basis for the research conducted at BISC the axiomatic system for Euclidean geometry as given by Hilbert in [15] has been chosen. The following tasks have been adressed:

- identifying suitable fuzzy interpretations of Hilbert's primitive geometric objects, namely *points* and *lines*.
- identifying suitable fuzzy interpretations of Hilbert's primitive geometric relations, namely *equality*, *incidence*, *betweenness* and *congruence*.
- identifying suitable approaches within the framework of fuzzy logic and approximate reasoning to modeling deductive systems.

An extensive literature review has been conducted during the whole period of the research visit, which was facilitated by the fact that Prof. Zadeh invented the theory of fuzzy logic at UC Berkeley in 1965. As a

result, the available material concerning this research field at the UC Berkeley library is overwhelming.
According to the aforementioned tasks, approaches to specifying fuzzy sets, fuzzy relations and fuzzy approximate deduction have been investigated and tested for their applicability to the given problem. On the basis of the reviewed literature, the course contents and the extensive discussions with the BISC group members and visitors during seminars and round table appointments, considerable progress has been made in identifying useful approaches and excluding unsuitable ones.

Overview:

During the first and second month, the main focus of research lay on a literature review of the fundamentals of *fuzzy logic in the narrow sense*, i.e. the branch of fuzzy logic dealing with mathematical multivalued logic. Among many other books and articles, the following books turned out to be particularly useful: [20], [11], [17], [14], [16], [19], [9]. A basic understanding of the field has been achieved and the different possible choices of algebaic structures that can underly a fuzzy logic have been reviewed.
In the course of the first attempts in modelling Hilbert's axiom schemata, it has become clear that the type of fuzzy sets that are used to model the primitives *point* and *line* necessarily result from the choice of fuzzy relations for modelling the *equality*, *incidence*, *betweenness* and *congruence*. For this reason, a further analysis of fuzzy sets that are suitable to model fuzzy geographic entities has been postponed.
During the third month, fuzzy relations have been investigated that can be used as candidates for fuzzifying the primitive geomtric relations *equality*, *incidence*, *betweenness* and *congruence* used in Hilberts axiomatization of Euclidean geometry. Since the uniqueness of geometric constructions is of special importance for all geometric reasoning, a focus has been put on modelling the *equlity* relation of geometric primitives.
In the fourth and fivth month, the emphasis has been put on *fuzzy logic in the broader sense*, i.e. approximate reasoning techiques such as generalized modus ponens, fuzzy if-then-rules, the theory of linguistic variables, fuzzy quantifyers, analogical reasoning, fuzzy control.

Identification of a suitable algebraic structure:

Łukasiewicz algebra has been identified to be most suitable for the description of geometric problems. The reason lies in the fact that Łukasiewicz algebras play a distinguished role for representing the algebra of truth values [17]. Another reason, which is geometrically motivated, is the fact that fuzzy indistinguishability relations can be shown to be dual to pseudo-distances, if the Łukasiewicz t-norm is used [4].

Fuzzy relations:

Fuzzy indistinguishability relations have been identified as being of particular importance for modelling geometric reasoning under uncertainty. Fuzzy indistinguishability relation are the fuzzy interpretations of crisp equivalence relations. As such, they can be used to fuzzify the *equality* of the geometric primitives *point* and *line*. Since in all geometric deductive reasoning, i.e. already in relatively simple incidence geometries, the uniqueness of geometric constructions is crucial, a special emphasis has been put on this point.

It turned out that for most semantical interpretations of geometric objects with uncertainty in location, *equality* is not transitive. This phenomenon is commonly known as the Poincaré paradox [18], [8], [3]. As a consequence, fuzzy indistinguishability relations are not suitable for modeling *equality* under positional uncertainty. *Resemblance relations* [8] or *approximate t-similarity relations* [12] must be employed.

Approximate reasoning:

The Mamdani type inference and the Takagi-Sugeno type inference for fuzzy control have been investigated [20] and could be excluded as suitable approaches to fuzzy deductive reasoning. The *fuzzy if-then-rules* used in fuzzy control are presented as implications, but, from the logical point of view, do not model the implication operator in a reasonable way. Additionally, “it has slowly become clear, that fuzzy control deals with *approximation of functions* on the basis of pieces of fuzzy information of the kind “for arguments approximately equal to c_i the image is approximately equal to d_i”. ” ([14], p.177)

In contrast to this, fuzzy if-then-rules can be interpreted as linguistically characterized logical implications, which receive their semantic interpretation and validation from expert knowledge. On this basis, deduction on simple linguistic descriptions is possible [17]. Based on this idea, the workshop-paper “Approximate Geometric Reasoning With Extended Geographic Objects” has been published (see appendix).

Another approach to approximate deduction is provided by *Rational Pavelka Logic (RPL)* [14]. In RPL, it is possible to draw pertially true conclusions from partially true premises. Whereas deduction on linguistic descriptions is based on partially true predicates, in RPL the axioms themselves are allowed to be partially true and (partially true) conclusions can be derived from them. A conference paper based on this idea is planned and in process ("Geometric Reasoning Under Uncertainty with Rational Pavelka Logic", see chapter 6.2).

Conclusions and outcome:

Fuzzy logic and approximate reasoning provide a fromal apparatus for fromulating deductive reasoning schemes under uncertainty. Its applicability to the problem domain of geometric uncertainty modelling for GIS could be confirmed and significant progress has been made towards formulating a fuzzified model of axiomatic Euclidean geometry. Instead of fuzzyfying the geometrical objects and operations on the object level only, the approximate reasoning framework provides techniques that allow to additionally fuzzify the logical inference mechanism that underlies the axioamtic system on a metalogical level.
It is expected that a significant subset of Euclidean gemetric reasoning can be fuzzyfied with this approach, namely the subset that is formalizable with first order logic. This subset comprises of all axioms of incidence, betwenness and congruence and ommits the archimedian property of continuity. Since geometric reasoning in a GIS setting is always restricted to a finite map frame, continuity to infinity is not a necessary condition.

6.2 Presentations and Publications

A presentation at the BISC Round Table has been given. One extended abstract for a doctoral colloquium and a workshop paper has so far been submitted and accepted. As a result from discussions with BSIC members and the research conducted at the BISC group, two publications are in process. In the following the presentation and the publications are listed:

Wilke, G., ***Positional Uncertainty in Geographic Information Systems***. Presented at the BISC Round Table Discussion on 06/05/2009. For a list of Round Table Discussions see chapter 5.2. For abstract and slides see appendix.

The scholarship holder's PhD topic, the motivation from the field of geographic information science an the relation to fuzzy logic has been addressed. The audience comprised of the BISC goup members and serveral visitors. The relevance of the work was confirmed by the audience.

Wilke, G., ***A Model of Positional Uncertainty for the Vector Data Model Based on Axiomatic Geometry***. Accepted for Publication at the COSIT 2009 doctoral colloquium. See appendix for the extended astract.

Wilke, G., ***Approximate Geometric Reasoning With Extended Geographic Objects***. Accepted for Publication at the *ISPRS-COST Workshop on Quality, Scale, and Analysis Aspects of City Models*, Lund, Sweden, Dec. 3-4, 2009. See appendix for full paper.

Working title: ***Pointless Pseudometric Spaces for Geographic Information Systems***. Planned for submission to the *International Symposium on Spatial Accuracy Assessment in Natural Resources and Environmental Sciences (Accuracy2010)*, Leicester, UK, July 20-23, 2009.

Working title: ***Geometric Reasoning Under Uncertainty with Rational Pavelka Logic***. Planned for submission to the *International Conference on Geographic Information Science* (AGILE), May 11-14, 2010.

6.3 Significance of results

The integration of techniques to handle positional uncertainty in GIS is a dominating research topic in geographic information science. Current vector-based GIS software is based on idealized geometric objects: Infinitely small points and infinitely thin lines disregard the real character of object representation. As a result, inconsistencies are created and inherently vague objects cannot be represented in a relasitic way.
Most of the ongoing research on positional uncertainty in the GIS community is concerned with modeling topological and metrical operations [21], [6], [2], [1], [7] whereas geometric operations are ommited. In GIS, geometrical operations occur in the majority of spatial analsysis operations such as line intersection, polygon overlay, point-in-polygon-analysis or buffer creation. For this reason, a geometric calculus for uncertain objects that parallels the usual crisp calculus of Cartesian geometry is highly desirable.

6.4 Future work

The research conducted at UC Berkeley is part of the scholarship holder's dissertation program. The PhD- research objective of formulating a fuzzified axiomatic deduction system for Euclidean geometry under uncertainty for GIS will be completed within the next 15 months.

In particular, *Rational Pavelka Logic* and *deduction with linguistic expressions* will be investigated in more detail. The properties of *approximate t-similarities* as models of the geometric *equality* relation and it's relation to the geometric primitive relations *incidence*, *betweenness* and *congruence* will be studied.

7 Summary

The objectives posed in chapter 4 have been achieved in large part:

- The scholarship holder gained considerable expertise in the field of fuzzy logic and approximate reasoning theory and application.
- The methodology of approximate reasoning has been applied successfully to parts of the problem domain. It is expected that the task can be completed within the coming academic year.
- A vivid echange of ideas took place during courses, Round Table Discussions and in private conversation. The scholarship holder is in close contact to members and visitors of the BISC group, first and foremost with Prof. Zadeh, Dr. Celikyilmaz (BISC) and Dr. DeCoensel (TU Gent). A connection was be established to the members of the Institute for Research and Applications of Fuzzy Modeling (Uni Ostrava), with whom a future collaboration is considered.
- In order to communicate the gained knowledge to collegues at the home department, a presentation will be given at the joint retreat of the Department of Geoinformation and Cartography (TU Vienna) and the MUSIL research group (Uni Münster) at at 09/28/2009 in Vienna.

8 Acknowledgments

I want to thank the Marshall Plan Foundation for making this research visit possible. In particular I want to thank Mag. Zemann and Mag. Stoiber for their friendly support with all administrative tasks.
At the University of California, Berkeley, I want to thank foremost and most warmly Prof. Lotfi Zadeh, who invited me to the BISC group and spend a lot of time listening to my research problems. I am especially obliged to my PhD advisor and supervisor Prof. Frank, who encouraged me to apply for the Marshall Plan Scholarship and helped me enormously with a myriad of little hints for the preparation for the trip.

The members of the BISC group welcomed me most warmly. Especially Dr. Celikyilmas spent a lot of time for discussions with me.

References

[1] Brandon Bennett, Anthony G. Cohn, Paolo Torrini, and Shyamanta M. Hazarika. Region-based qualitative geometry. Research report series, University of Leeds, School of Computer Studies, 2000.

[2] Thomas Bittner and John Stell. Vagueness and rough location. *GeoInformatica*, 6(2):99–121, 2002.

[3] Ulrich Bodenhofer. A note on approximate equality versus the poincar paradox. *Fuzzy Sets and Systems*, 133(2):155–160, 2003.

[4] D. Boixader. On the relationship between t -transitivity and approximate equality. *Fuzzy Sets and Systems*, 133(2):161–169, 2003.

[5] James D. Brown and Gerard B.M. Heuvelink. The data uncertainty engine (due): A software tool for assessing and simulating uncertain environmental variables. *Computers and Geoscience*, 33:127–190, 2007.

[6] Eliseo Clementini. A model for uncertain lines. *Journal of Visual Languages & Computing*, 16:271–288, 2005.

[7] Anthony G. Cohn and N. M. Gotts. *Geographic Objects with Indeterminate Boundaries*, chapter The 'Egg-Yolk' Representation od Regions with Indeterminate Boundaries, pages 171–188. Number 2 in GISDATA. CRC Press, 1996.

[8] Martine De Cock and Etienne Kerre. On (un)suitable fuzzy relations to model approximate equality. *Fuzzy Sets and Systems*, 133(2):137–153, 2003.

[9] Didier Dubois and Henri Prade. *Fuzzy Sets and Systems: Theory and Application*. Acadeic Pr Inc, 1980.

[10] Matt Duckham, Keith Mason, John Stell, and Mike Worboys. A formal approach to imperfection in geographic information. *Computers, Environment and Urban Systems*, 2001:89–103, 2001.

[11] Giangiaccomo Gerla. *Fuzzy Logic - Mathematical Tools for Approximate Reasoning*, volume 11 of *Trends in Logic*. Kluwer Acadeic Publishers, 2001.

[12] Giangiaccomo Gerla. Approximate similarities and poincare paradox. *Notre Dame Journal of Formal Logic*, 49(2):203–226, 2008.

[13] Michael F. Goodchild. Measurement-based gis. In W. Shi, P.F. Fisher, and M.F. Goodchild, editors, *Proceedings of the International Symposium on Spatial Data Quality 99*, pages 1–9. Polytechnic University, Hong Kong, 1999.

[14] Petr Hajek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer Academic Publishers, 1998.

[15] David Hilbert. *Grundlagen der Geometrie*. Teubner Studienbuecher Mathematik, Stuttgart, 9 edition, 1962.

[16] George J. Klir and Bo Yuan. *Fuzzy Sets and Fuzzy Logic - Theory and Applications*. Prentice Hall, 1995.

[17] Vilem Novak, Irina Perfilieva, and Jiri Mokor. *Mathematical Principles of Fuzzy Logic*. Kluwer Academic Publishers, 1999.

[18] Henri Poincare. *Science and Hypothesis*. Walter Scott Publishing, 1905.

[19] Da Ruan and Chongfu Huang, editors. *Fuzzy Sets and Fuzzy Information-Granulation Theory - Key Selected Papers by Lotfi A. Zadeh*. Beijing Normal University Press, 2000.

[20] Enrique H. Ruspini, Piero P. Bonissone, and Witold Pedrycz, editors. *Handbook of Fuzzy Computation*. IOP Publishing Ltd, October 1998.

[21] Steven Schockaert. Modelling nearness and cardinal directions between fuzzy regions. In *IEEE World Congress on Computational Intelligence*, 2008.

[22] Lotfi A. Zadeh. Outline of a new approach to the analysis of complex systems and decision processes. *IEEE Transactions on Systems, Man and Cybernetics*, 1:28–44, January 1973.

[23] Lotfi A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning - I. *Information Sciences*, 8:199–249, 1975.

[24] Lotfi A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning - II. *Information Sciences*, 8:301–357, 1975.

[25] Lotfi A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning - III. *Information Sciences*, 9:43–80, 1975.

9 Appendix

9.1 Presentation: *Positional Uncertainty in Geographic Information Systems*

9.2 Doctoral Colloquium: *A Model of Positional Uncertainty for the Vector Data Model Based on Axiomatic Geometry*

9.3 Research Paper: *Approximate Geometric Reasoning With Extended Geographic Objects*

Positional Uncertainty in Geographic Information Systems

Abstract:

Geometric functionality in vector based geographic information systems (GIS) is based on infinitely small points and infinitely thin lines. This is in contrast to the fact that geographic features and their representation are extended and uncertain in location. Object representation in GIS often destroys the consistency of geometric reasoning.

The existing geometric models of positional uncertainty for vector based GIS are not implemented in current GIS software. The reason lies in the fact that they do not allow geometric constructions analogous to crisp constructions: the pertinent operations are either not defined or not algebraically closed.

To overcome these difficulties an axiomatic approach is proposed that uses fuzzy logic models of Hilbert's axiomatization of Euclidean Geometry. The axiomatic approach ensures the consistency of geometric reasoning despite the uncertainties in the locations.

Positional Uncertainty in Geographic Information Systems

- GIS Data Models
- Spatial Analysis in Vector Based GIS = Geometry
- Positional Uncertainty in Vector Based GIS
- Topologic-Metric Models
- Geometric Models
- Hypothesis
- Axiomatic Geometry
- Fuzzy Models

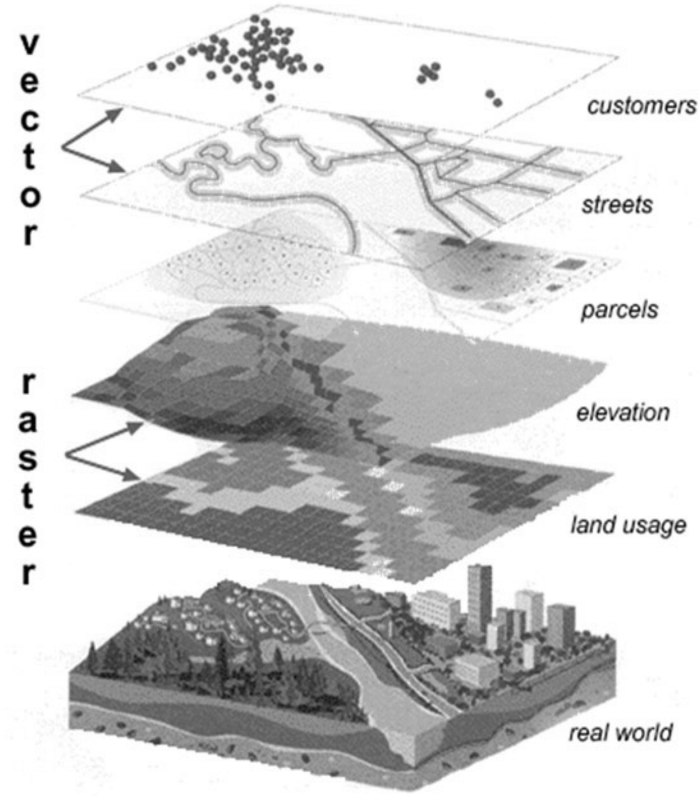
GIS Data Models

<u>Data</u>: from surveying
<u>Usage</u>: socially constructed geographic objects

Parcel boundaries, postal zones, census tracts, electric underground cable network, water & gas pipeline network, street networks, public transportation, tourist information system

<u>Data</u>: satellite & air borne imagery
<u>Usage</u>: natural phenomena

Land cover, land usage, elevation, soil types, environmental modeling, e.g. pollution risk assessment, prediction of wildfire dissemination, impact analysis, simulations



Spatial Analysis in Vector Based GIS = Geometry

<u>Geom. data types:</u>

<u>Layers:</u>

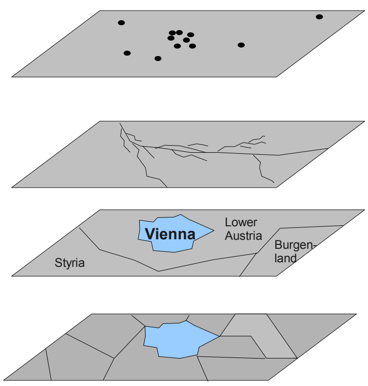
<u>Themes</u>, e.g

points = real-world coordinates (long/lat) after projection — emergency coordination center

lines segments = ordered pairs of points — streams

polygones = closed ordered n-tupels of line segments — county lines

polygones — emergency service zones



2 Databases:

Geometry Topology ⟷ (object-ID) ⟷ Attributes (relational db)

Spatial Analysis in Vector Based GIS = Geometry

Typical spatial queries:

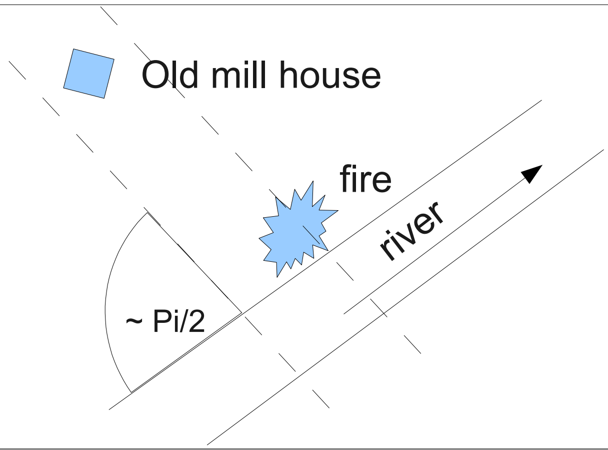
- How many of the national emergency coordination centers are located in Vienna?
  (**geometric query**: needs coordinates)
- What is the fastest way from Vienna city center to "Wiener Neustadt"?
  (**network query**: network topology & weights)
- How many inhabitant per emergency service zone?
  (geometric query)
- How many first class hotels are within a 1000m radius from the Vienna opera house?
  (**geometric query**)

- How many pubs are in **close** walking distance from the **Bermuda triangle**?
- How many guesthouses are **between** the **archaeological excavation** Canuntum and **Bratislava**?
- The fire is **on the left bank** of the river, **about level with** the **old mill house**.

Positional Uncertainty in Vector Based GIS

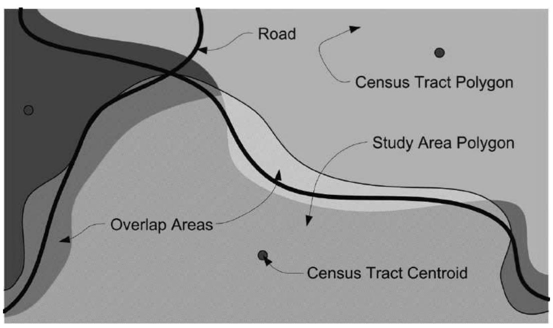
Verbal reports

"The fire is **on the left bank** of the river, **about level with** the **old mill house**."



Combination of different data sources with different lineage

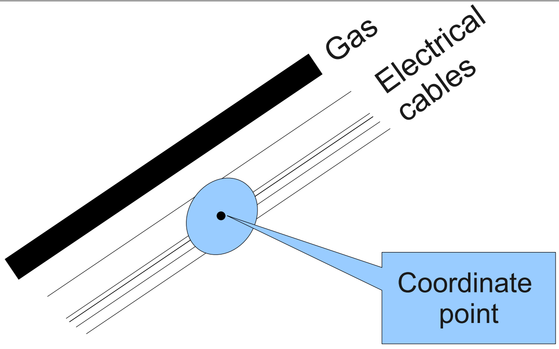
Hand digitizing errors



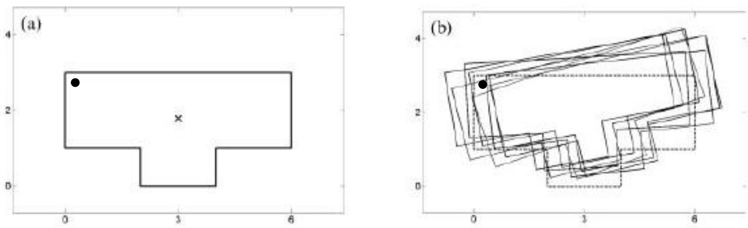
Positional Uncertainty in Vector Based GIS

Measurement Errors

Does the company hit the gas pipeline when digging for the electrical uncerground cable?



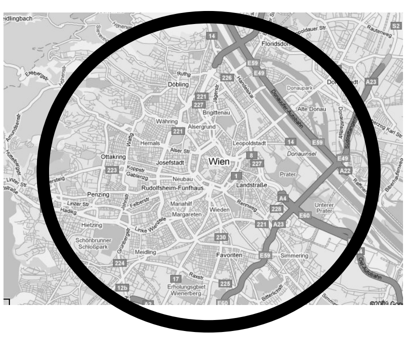
Is the point within the polygon?

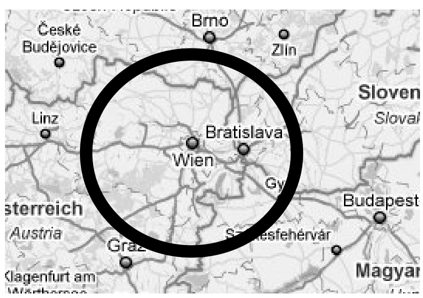


Positional Uncertainty in Vector Based GIS

Cartographic generalization

In a large scale map the extend of the representation of Vienna is not (much) related with the real extend.





Vienna in different map scales

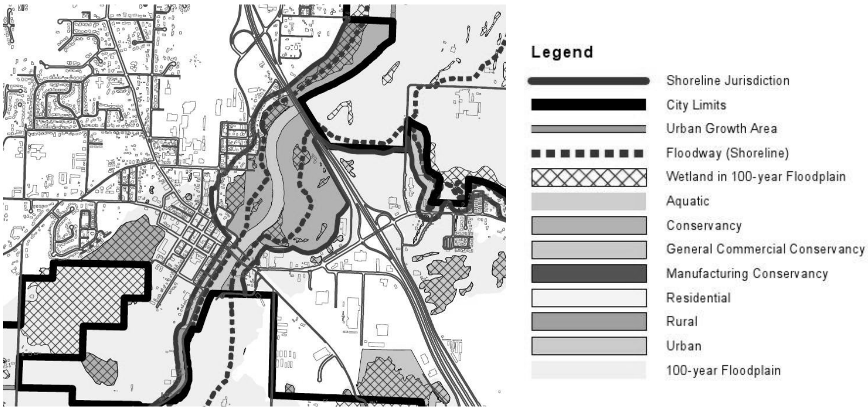
Positional Uncertainty in Vector Based GIS

Inherently uncertain objects
(natural phenomena)

Boundaries of lakes often have transition zones.

Dynamic spatial phenomenon
(natural phenomena)

Is my house in the 100- year flooding area?

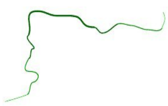


Topological-Metrical Model 1:
Vage Regions (fuzzy sets)

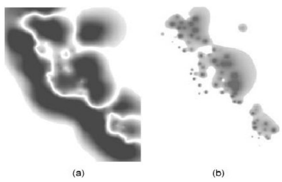
A. Dilo (TU Delft), A. Stein, (ITC Enschede), (2006)

Objects:
fuzzy sets with
support = open set

0D fuzzy set



1D fuzzy set (parameterized, topology = relative topology)



2D fuzzy sets

Representation in VDM:
surface, i.e. TIN

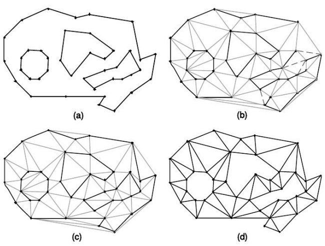


Fig. 13. Results of three steps constrained Delaunay triangulation in a simple vague region. **(a)** region boundaries, **(b)** Delaunay triangulation, **(c)** insertion of missing boundary lines, **(d)** removal of triangles outside the boundary and inside holes.

Top. Operations: set-intersection, set-join, closure, interior, difference, distance, area, etc.

Difference of (a) and (b)
= fuzzy difference

$$\mu \sqcap \left(1_{\mathbb{R}^2} - \nu \right)$$

Topological-Metrical Model 2:
Objects with Broad Boundaries

E. Clementini (Univ. of L'Aquila, Italy), (2005)

Objects:
set with 3-valued
domain

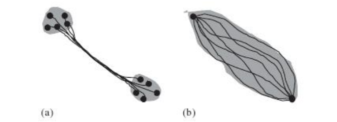


Fig. 2. Broad boundary (a) and broad interior (b) of a simple line.

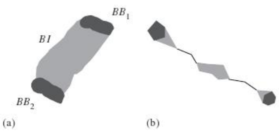


Fig. 1. A region with a broad boundary.

Region with
broad boundary

Representation in VDM:
9-intersection model
(relation algebra)

$$M = \begin{pmatrix} L_1^\circ \cap L_2^\circ & L_1^\circ \cap \Delta L_2 & L_1^\circ \cap L_2' \\ \Delta L_1 \cap L_2^\circ & \Delta L_1 \cap \Delta L_2 & \Delta L_1 \cap L_2' \\ L_1' \cap L_2^\circ & L_2' \cap \Delta L_2 & \bar{L}_1 \cap L_2' \end{pmatrix}$$

Matrix has boolean values
(empty -and non-empty)

Top. Relations: touches,
intersects, overlaps, crosses, in, etc.

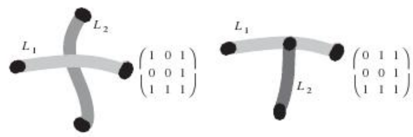
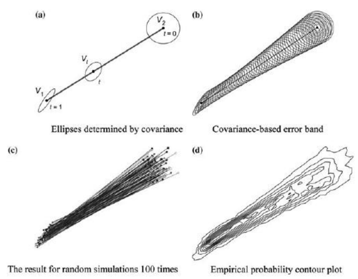


Fig. 11. Examples of topological relations between uncertain lines expressed by the 9-intersection.

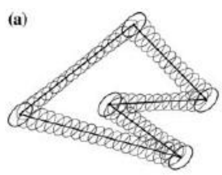
Geometric Model 1:
Covariance based error band model

Yee Leung (Univ. Hong Kong), Ma (Chang'an Univ.), M. Goodchild (UCSB):

Objects:
set with 3-valued
domain



Points and lines



Polygon

- Algebraically closed (line intersection)
- Point-in-polygon solved
- Polygon-on-polygon solved

Metric proposed (distance, area)

BUT: isomorphic to crisp geometry. incidence is not a gradual (probabilistic) concept

Geometric Model 2:
Fuzzy geometry

Buckley, Eslami (1997):

Method 2. A fuzzy point at (a,b) in R^2, written $\bar{P}(a,b)$, is defined by its membership function:
1. $\mu((x,y)|\bar{P}(a,b))$ is upper semi-continuous;
2. $\mu((x,y)|\bar{P}(a,b)) = 1$ if and only if $(x,y)=(a,b)$; and
3. $\bar{P}(\alpha)$ is a compact, convex, subset of R^2 for all α, $0 \leqslant \alpha \leqslant 1$.

Method 6 (*Two-point form*). Let $\overline{P_1}$ and $\overline{P_2}$ be two fuzzy points in the plane. Define

$$\Omega_3(\alpha) = \left\{(x,y) : \frac{y-v_1}{x-u_1} = \frac{v_2-v_1}{u_2-u_1}, (u_1,v_1) \in \overline{P_1}(\alpha), (u_2,v_2) \in \overline{P_2}(\alpha)\right\}, \quad \text{for } 0 \leqslant \alpha \leqslant 1.$$

$\overline{L_3}$ is

$$\mu((x,y) \mid \overline{L_3}) = \sup\{\alpha : (x,y) \in \Omega_3(\alpha)\}.$$

Definition 4. We say a fuzzy line $\bar{L}$ contains a fuzzy point $\bar{Q}$ if and only if $\bar{Q} \leqslant \bar{L}$.

Fuzzy lines are not unique; No fuzzy line-intersection defined; etc.

Research Hypothesis

"It is possible to define an axiomatic model of 2D geometry that incorporates geometric operations with tolerance for positional uncertainty. These operations apply to objects that reflect the extended character of geographic features and the positional uncertainty in their representation in GIS."

Idea: Axiomatic f-Geometry for GIS

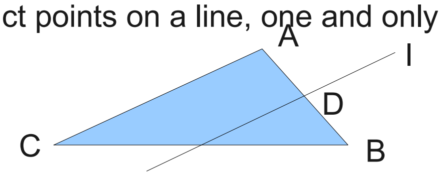
Basis: Hilbert's axiomatic system for Euclidean Geometry:

Axiom Set 1: Incidence

- 2 points define at least one line
- 2 points define at most one line
- a line contains at least 2 points
- three non-collinear points exist

Axiom Set 1: Betweenness:

- If B is between A and C (A*B*C), then A,B,C are distinct collinear points.
- For distinct points A,B, a point C exists s.th. A*B*C.
- Given 3 distinct points on a line, one and only one of them is between the other two.
- (Pasch)



A,B,C non-collinear points

Axiom Set 3: Congruency

Axiom Set 4: Parallelism.

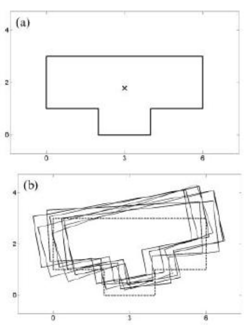
Idea: Axiomatic f-Geometry for GIS

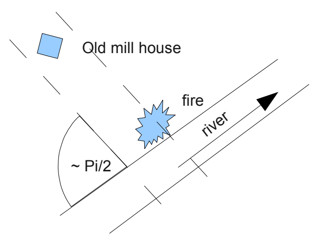
Uncertainty representation of points and lines segments such that

1) Both are “extended” (topologically neighbourhoods)

2) Lines can be “derived” from points, i.e. two points determine a line

3) Behave like points and lines, e.g. two lines intersect in a point





Fuzzy Models

Pavelka style approximate reasoning:

Graded Inference Rules:

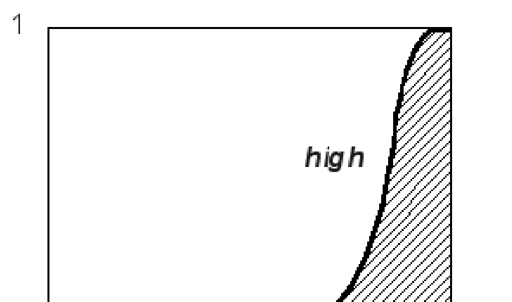
$$\frac{\alpha_1,\ldots,\alpha_n}{r'(\alpha_1,\ldots,\alpha_n)} \quad ; \quad \frac{\lambda_1,\ldots,\lambda_n}{r''(\lambda_1,\ldots,\lambda_n)} \quad .$$

IF you know that the formulas $\alpha_1,\ldots,\alpha_n$ are true (at least) to the degree $\lambda_1,\ldots,\lambda_n$,
THEN you can conclude that $r'(\alpha_1,\ldots,\alpha_n)$ is true (at least) to degree $r''(\lambda_1,\ldots,\lambda_n)$.

Fuzzy Models 2

Deduction from Linguistic Variables:

IF sep(P,Q) is high
THEN
inc(P ∪ Q ,P) is high AND
inc(P ∪ Q ,Q) is high AND
lSh(P ∪ Q) is high.
IF inc(S',P) is high AND
inc(S',Q) is high AND
lSh(S') is high
THEN
eq(P ∪ Q ,S') is high.



A Model of Positional Uncertainty for the Vector Data Model Based on Axiomatic Geometry

Gwen Wilke

Vienna University of Technology, Department of Geoinformation and Cartography,
Gusshausstrasse 27-29 E127, A-1040 Vienna, Austria
wilke@geoinfo.tuwien.ac.at
http://www.geoinfo.tuwien.ac.at/

Key words: Geographic information systems, imprecise spatial knowledge, positional uncertainty, axiomatic geometry, vector data model, fuzzy set theory.

1 Positional Uncertainty of Vector Data

An important research topic in geographic information science is the incorporation of positional uncertainty in geographic information systems (GIS). The field based data model can easily handle uncertainty in location. Vector based systems lack a natural representation of positional uncertainty, because their conceptualization is based on mathematically ideal objects like infinitely small points and infinitely thin lines.

Two main approaches for incorporating positional uncertainty of point and line features for the vector data model exist: (1) buffer based models, like e.g. the error band model [13], represent statistical variations due to measurement errors; (2) objects with broad boundaries represent either statistical or vague data in a framework of a 3-valued indeterminacy [3]. Both approaches fail to implement the whole range of tests and spatial operations that can be employed within exact Cartesian geometry. The reason lies in the fact that uniqueness of representation of points and lines is given up, resulting in ambiguous, ill-defined or contradictory geometric constellations of some of their exact realizations. One example is the ill-defined point-in-polygon test as described in [8] and [5].

The goal of the present work is to disambiguate these situations by treating objects with uncertainty in location as entireties instead of looking at their exact realizations. A geometric model is defined that operates on points and lines with uncertainty in location. The primitive objects and relations of the model are defined in a way that complies with the axiomatic system for Euclidean geometry given by David Hilbert in 1899 [6]. The axiomatic approach guarantees consistency of the model: every interpretation of the geometric primitives *point*, *line*, *incidence*, *betweenness*, *congruency* and *parallelism* that complies with Hilbert's axioms must be free of contradictions. The formal apparatus provided by the axiomatic system allows to derive the whole range of composite figures, geometric operations and tests used in vector based GIS, because all theorems of Euclidean geometry remain valid within the model.

The approach proposed in the present work focuses on Euclidean plane geometry. The primitive objects *point* and *line* of Hilbert's axiomatic system are modeled by connected regions in $\mathbb{R}^2$. Primitive relations operate on the point models and line models as entireties. They are algebraically closed over the domain of objects with uncertainty in location and satisfy the axioms of Euclidean geometry. The axiomatic approach guarantees that the whole range of geometric operations and tests of vector based GIS can be derived without contradictions. The new contribution of this work is the definition of a valid model of Euclidean geometry that is based on objects with uncertainty in location.

2 Research Hypothesis

A point with positional uncertainty cannot be unambiguously mapped to a single coordinate point in the Cartesian space. Its representation must involve a subset of $\mathbb{R}^2$, respectively $\mathbb{R}^3$, that consists of more than one coordinate point. The present work focuses on two dimensional geometric models embedded in the Cartesian plane. It is assumed that the set of coordinate points representing the possible locations of a point with positional uncertainty can be described by a connected region in $\mathbb{R}^2$. As a consequence the geometric model to be defined is a calculus of regions.

The hypothesis of the present work states that a valid axiomatic model of Euclidean plane geometry for vector-based GIS can be defined that represents points and lines with uncertainty in location on the basis of connected regions embedded in the Cartesian plane.

3 Testing Existing Models for Validity

The model building process consists of two steps: In a first step existing approaches to modelling positional uncertainty and approaches to building geometries of extended objects in GIS and in other research fields are listed. These approaches are analyzed for their definitions of primitive objects, relations and operations. In a second step the listed models are tested for validity within the axiomatic system of Euclidean geometry defined by Hilbert [6].

Hilbert's axiomatic system consists of two primitive objects (*points* and *lines*) and four primitive relations (*incidence*, *betweenness*, *congruency* and *parallelism*). All primitives remain undefined and are only characterized by their properties, given by the axioms. The background logic of the axiomatic system additionally employs an *identity relation* between points and lines, respectively. Every *interpretation* of the primitives that satisfies the axioms is called a valid *model* of Euclidean geometry.

Axioms are grouped by the type of primitive relation they invoke. Since each group of axioms builds upon the foregoing groups, models can be tested in a step by step process, starting with the axioms of incidence. If a model does not provide all necessary primitives, a consistent interpretation is given, if possible. If the interpretation of primitives do not satisfy one or more axioms of a group, the reason is analyzed. During the course of analysis it becomes increasingly clear which criteria a region-based model of points and lines with positional uncertainty must satisfy in order to comply with certain groups of axioms. According to these criteria a new model is formulated.

4 Intermediate Results: Towards a Fuzzy Model

The following approaches have been identified to be promising candidates for defining an axiomatic model of Euclidean geometry for objects with positional uncertainty which is based on regions: the *Covariance-Based Error Band Model* for measurement-based GIS by Leung et. al. [7], the model of *Objects With Broad Boundaries* by Clementini [3], *Fuzzy Geometry* defined by Buckley and Eslami [2], *Geometry for Places* by Schmitdke [12], *Region-Based Geometry* formulated by Bennett et. al. [1] and *Pointless Geometry* by Gerla [4].

The first three of these approaches have been tested for compliance with the axioms of incidence. All three approaches define models for points and lines with uncertainty in location, but only one of them explicitly defines an identity and an incidence relation. Several interpretations of identity and incidence relations have been tested together with the respective models. In all cases the models already fail to satisfy Hilbert's very first axiom, which postulates uniqueness of line representation: in general two distinct points with positional uncertainty define more than one line with positional uncertainty that is incident with them.

The reason for this outcome lies in the fact that in the presence of positional uncertainty the identity relation of the background logic formally translates into a bivalent or interval-valued indiscernibility relation on the set of points with uncertainty in location, depending on the model and it's semantic. As soon as we adopt the existential part of Hilbert's first incidence axiom - "If A and B are distinct points, *there is at least one line* that is incident with both A and B" [10] - the indiscernibility relation induces an ambiguity in the line representation. This outcome is in accordance with intuition and a desirable feature when modelling positional uncertainty.

As a consequence, we must ask the question, if it is possible to embed one of the existing models in a richer model and, by adding information, restoring the uniqueness of line representation, while at the same time the desirable ambiguity in the restricted model is maintained. The approach has the advantage that all theorems of Euclidean geometry remain valid within the richer model and existing GIS-operations can be reused [11]. To obtain a result in the restricted model, operations are performed in the richer model and the result is restricted afterwards.

A promising modelling language for this endeavor is fuzzy set theory and fuzzy logic, which is built upon the notion of indiscernibility [14]: The richer model can be formulated in terms of membership functions, the restrictions are given by α-cuts, where the value of α may determine the degree of discernibility. The language of fuzzy set theory does not exclude probabilistic interpretations of membership functions [9]. The formulation of such a model and the applicability of fuzzy set theory to modelling measurement error is subject of ongoing research.

5 Further work

Based on a valid model of geometric primitives with uncertainty in location the properties of derived geometric concepts like angle, line segment, ray, area, orientation and symmetry will be investigated. E.g. the notions of incidence and betweenness are sufficient to derive the concept of orientation.

In order to measure distances, a metric or a *weak metric*, as for instance proposed by Buckley and Eslami [2], will be defined on the set of points with uncertainty in location. The potential of a weak metric to induce a topology will be investigated.

The model of Euclidean plane geometry for points and lines with positional uncertainty will be extended to three dimensions: Planes with uncertainty in location will be defined as a third primitive object of the model.

6 Acknowledgement

The research was funded by the Austrian Marshall Plan Foundation.

References

1. Bennett, B.; Cohn, A. G.; Torrini, P., Hazarika, S. M.: A Foundation for Region-Based Qualitative Geometry. In: Proceedings of ECAI-2000, pp. 204–208, (2000)
2. Buckley, J., Eslami, E.: Fuzzy plane geometry I: Points and lines. Fuzzy Sets and Systems 86, 179–187 (1997)
3. Clementini, E.: A model for uncertain lines. Journal of Visual Languages and Computing 16, 271–288 (2005)
4. Gerla, G.: Pointless geometries. In: Buekenhout, F., Kantor, W. (eds.). Handbook of Incidence Geometry, pp. 1015–1031. North-Holland (1995)
5. Heuvelink, G., Brown, J., van Loon, E.: A probabilistic framework for representing and simulating uncertain environmental variables. IJGIS 21, 497–513 (2007)
6. Hilbert, D.: Grundlagen der Geometrie. Teubner Studienbücher Mathematik, Stuttgart (1968)
7. Leung, Y., Ma, J., Goodchild, M. F.: A general framework for error analysis in measurement-based GIS Part 1: The basic measurement-error model and related concepts. J. Geograph. Syst. 6, 325–354 (2004)
8. Leung, Y., Ma, J., Goodchild, M. F.: A general framework for error analysis in measurement-based GIS Part 2: The algebra-based probability model for point-in-polygon analysis. J. Geograph. Syst. 6, 355–379 (2004)
9. Krishnapuram, R.: Membership function elicitation and learning. In: Ruspini, E. H., Bonissone, P. P., Pedrycz, W. (eds.) Handbook of Fuzzy Computation pp. B3.2:1–B3.2:11. IOP Publishing Ltd, (1998)
10. Meserve, B. E.: Fundamental Concepts of Geometry. Dover Publications Inc. (1983)

11. Navratil, G., Karimipour, F., U.Frank, A.: Lifting Imprecise Values. In: The European Information Societry - 11th AGILE International Conference on Geographic Information Science, pp. 79–94. Springer (2008)
12. Schmidtke, H.: A geometry for places: Representing extension and extended objects. In: Kuhn, W., Worboys, M.F., Timpf, S. (eds.) Spatial Information Theory. LNCS, vol. 4128, pp. 1148–1158. Springer, Heidelberg (2006)
13. Shi, W.: A generic statistical approach for modelling error of geometric features in GIS. IJGIS 12, 131–143 (1998)
14. Zadeh, L. A.: Fuzzy sets and information granularity. In: Gupta, M., Ragade, R., Yager, R. (eds.) Advances in Fuzzy Set Theory and Applications, pp. 3-18. North-Holland Publishing Co., Amsterdam (1979a)

APPROXIMATE GEOMETRIC REASONING WITH EXTENDED GEOGRAPHIC OBJECTS

Gwen Wilke

Institute for Geoinformation and Cartography, TU Vienna, 1040 Vienna, Austria - wilke@geoinfo.tuwien.ac.at

KEY WORDS: Uncertainties in spatial data, qualitative spatial reasoning, fuzzy logic, axiomatic geometry, geographic information systems.

ABSTRACT:

The article presents a conceptual framework for formal geometric reasoning with extended objects in the context of vernacular geography. While point data is used to construct line features, polygon features and geometric operations for spatial overlay analysis, there is no canonical way of constructing a linear feature or polygon feature from extended objects such as vernacular place names or landmarks. Vernacular place names and landmarks are often used in tourist information brochures to describe the approximate location of hotels, restaurants or sites of interest. Place descriptions such as "The hotel is located in the centre between the opera house, the Louvre museum and the Champs-Elysees" cannot be queried in common geographic information systems (GIS). Due to differences in size, shape and scale, extended objects may cause ill-defined geometric constellations that cannot be used for geometric spatial analysis.

We propose to use fuzzy approximate reasoning techniques to keep track of the well-defined or ill-defined character of a geometric construction process. As an underlying deductive calculus for the reasoning process we use David Hilbert's axiomatic approach to geometric reasoning: based on the geometric primitives "extended point" and "extended line" geometric constructions can be deduced from a small set of axioms. Fuzzy reasoning has the advantage of being computationally less expensive and thus faster than an exact description of the model.

1. INTRODUCTION

1.1 Spatial analysis with extended objects

In colloquial speech, place names and landmarks are often used to describe the approximate location of geographic entities. For example, the statement “The hotel is located between the opera house, the Louvre museum and the Champs-Elysee” gives a qualitative description of the location and might be found in a tourist information brochure of Paris. Up to date, geographic information systems (GIS) have the ability to perform topological reasoning with extended geographic objects like “the Louvre museum” or “the Champs-Elysee”. Yet, the capability of geometric reasoning with extended objects is still missing. The aim of the present work is to lay a foundation for geometric reasoning with extended objects that is usable for GIS.

As an example of a geometric construction with extended objects consider an overlay analysis, where the approximate triangle formed by the opera house, the Louvre museum and the Champs-Elysee is intersected with another extended object, that approximately forms a rectangle. From a topological viewpoint, the interesting features of the resulting regions are if they are open, closed, one dimensional or two dimensional. From a geometric viewpoint, it is of interest if they are usable for further geometric constructions. As another example Figure 1 shows an approximate and ad hoc construction of the approximate location of the circumcenter of the same three objects.

In vector based GIS many tests and operations that are used for spatial analysis are based on the algebra of Cartesian coordinate geometry, and thus on the rules of Euclidean geometry. In order to extend geometric tests and operations to extended objects, it is necessary to look at the underlying rules of Euclidean geometry.

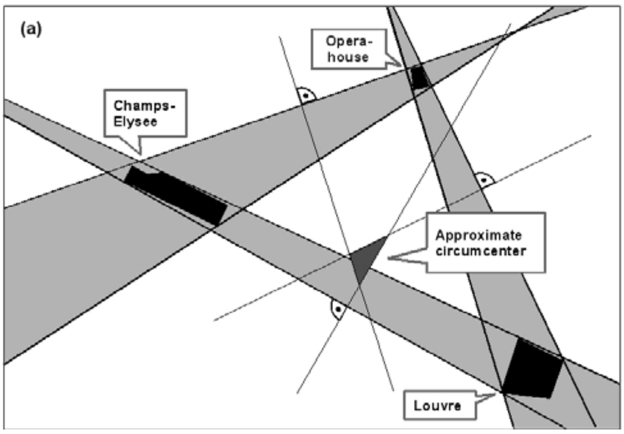


Figure 1. Approximate triangle formed by the Louvre, the Opera house and the Champs-Elysee in Paris: An approximate circumcenter can be derived.

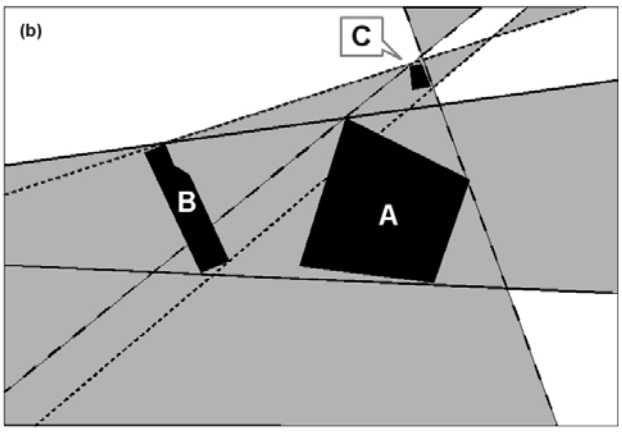


Figure 2. Ill-defined constellation: Connecting the extended objects A,B,C hardly makes sense. The construction of an approximate circumcenter is not meaningful.

In an abstract logical sense, Euclidean geometry can be seen as a calculus that is used to derive statements about geometrical entities from given facts. For example "The median lines of every triangle intersect in a unique point." is a true statement in Cartesian coordinate geometry. It can be logically derived from few basic facts, the axioms of Euclidean geometry. If we try to derive the same statement for triangles that are constructed from extended objects instead of constructing it from extensionless coordinate points, we may encounter difficulties: Some geometric constellations allow the derivation of meaningful geometric statements (Figure 1), others do not (Figure 2).
The present paper shows that the usefulness of a specific geometric constellation for further geometric reasoning is a matter of degree. We propose a framework for assigning grades of validity to geometric statements about extended objects that can be propagated through the steps of a geometric construction. As a first step, the present work looks at the incidence axioms of Euclidean geometry.
The remainder of the article is structured as follows: Chapter 2 briefly introduces the incidence axioms of Hilbert's axiomatic system; Examples of possible interpretations of non-extensionless geometric primitives are given; some arising problems are illustrated and formalized. Chapter 3 proposes graded truth values as a solution and the framework of approximate reasoning is introduced in the context of the paper. The article concludes with a discussion and with an outlook to further work.

1.2 Related Work

Most of the literature on qualitative spatial reasoning in the context of GIS is either topological or metrical in nature (Freksa, 1991; Frank, 1992; Dilo, 2006; Renz and Nebel,2007). Many of these approaches use fuzzy theory to represent uncertain or incomplete knowledge. It is very rarely the case that fuzzy logic is utilised as an approximate resoning technique.
One of the latter approaches has been introduces by S. Dutta (1990) for geometric and metric concepts. Dutta uses fuzzy approximate reasoning for the propagation of positional, metrical, propositional and range constraints through the process of geometric reasoning. It is conceptually similar to the present work, but does not develop a systematic approximate calculus based on axiomatic geometry. H. Schmidtke (2005), provides an axiomatic geometric approach to spatial reasoning, but focuses more on granularity than on geometry.
There are numerous approaches to defining Euclidean solid Geometry starting with the primitive notion of *region* or *sphere* instead of *point* (Tarski, 1956; Schmidt, 1979; Gerla, 1990, Bennett et. al., 2000). These approaches aim at restoring Euclidean geometry, including the concept of a crisp point, starting from different primitive objects. As a consequence, they are necessarily based on different primitive relations and operations than the ones commonly used in axiom systems such as (Hilbert, 1962). In contrast to this, the present work aims at applying the classical operations to extended objects. GIS users should be provided with the classical tools of spatial analysis that are well known and can be used without learning new and fundamentally different theories.

2. AXIOMATIC GEOMETRY AND EXTENDED OBJECTS

2.1 Geometric primitives and incidence

Euclidean geometry in its axiomatic form was introduced by Euclid in 300BC in his famous book Elements. In 1899 David Hilbert gave a complete and consistent formulation of an axiomatic system of Euclidean geometry (Hilbert 1962). The primitive objects in the two dimensional version of his formulation are points and lines. The most basic primitive relation between points and lines is the "on-relation", usually called incidence. The following four axioms formulate the behaviour of points and lines with respect to incidence:

1. For every two distinct points p and q, there exists a line l that is incident with p and q.
2. l is unique.
3. Every line is incident with at least two points.
4. There exist at least three points that are not incident with the same line.

The first and second axioms together state that it is always possible to connect two points with a unique line. In case of coordinate points p, q Cartesian geometry provides a formula for constructing this line: The parametric form reads

$$l = \{ p + t(q-p) \mid t \in \mathbb{R} \} . \qquad (1)$$

When we want to connect two extended geographic objects in a similar way, there is no canonical way of doing so. We can not refer to an existing model like Cartesian algebra. Instead, a new way of connecting extended objects must be found that respects the uniqueness property of axiom 2. In the following we will show that such a definition cannot be found without imposing too restricting conditions on the extended objects to be useful in a GIS context.

2.2 Connecting extended points

In the following we will refer to extended objects that play the geometric role of points as "extended points". As stated by axioms 1 and 2 of section 2.1, it should be possible to connect two extended points to form an "extended line". We will denote coordinate points and lines of Cartesian geometry by "Cartesian points" and "Cartesian lines". The meaning of the connection operation and the incidence relation for extended points and lines is a matter of definition. We will refer to such a definition as an "interpretation" of the connection relation and the incidence relation, respectively.
For extended geographic objects that are disc-shaped and of same size and that do not overlap, it is easy to give an interpretation of connection and incidence that complies with the axioms 1-4 of section 2.1 (Figure 3a): The connection of the extended points P and Q is interpreted as a minimal incident parallel stripe S, i.e. a stripe that is bounded by two parallel Cartesian lines, that is incident with both, P and Q, and has minimal width. Here, S is interpreted to be incident with P, if $P \cap S = P$. It is clear that the infimum over the width of possible parallel incident stripes exist and thus that S is unique. The third axiom holds trivially, and the fourth axiom holds whenever P and Q are proper subsets of the workspace. In case that P and Q are disc-shaped, but of different size, the above interpretation loses the uniqueness property (Figure 3b).

Figures 3c-3f sketch different possibilities to restore uniqueness by changing the interpretation of connection. In all cases, there seems to be a trade-off between uniqueness and usefulness for GIS purposes: In Figure 3c, connection is interpreted as the Cartesian line connecting the centres of gravity of *P* and *Q*. In this case all information on shape and size of *P* and *Q* is lost. In Figure 3d, connection is interpreted as the convex hull of the Cartesian point sets *P* and *Q*. In this case, the continuation of *S* to the left of *P* and to the right of *Q* is not defined. This leaves every intersection outside the convex hull undefined. Figures 3e and 3f propose two possibilities of continuation of the convex hull. Both variants impose additional constraints on *S* that are not derived from the data. These artificially added constraints create new constraints on subsequently constructed objects. For instance, the extended point *Q’* in Figure 3e is a translation of *Q* in the “main direction” of S. Intuitively, *Q’* should be incident with *S*, which it is not. Only if we shrink *Q’* to a Cartesian point, the incidence relation is satisfied.

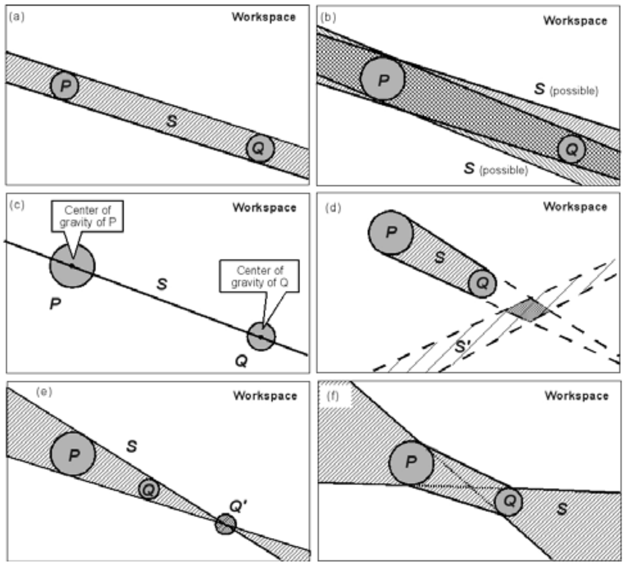


Figure 3. Different interpretations of the connection of two extended points.

In case we additionally drop the condition that extended points should be disc-shaped or if we allow them to overlap, the different interpretations of connection can become even less useful: Figure 4 shows two constellations where the connection of P and Q seems to result in a new extended point rather than in an object that represents an extended linear feature.

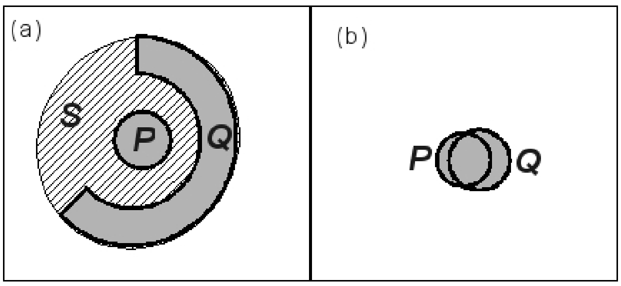


Figure 4. Convex-hull interpretation of the connection of two extended points (a) for arbitrary shapes, (b) for overlapping Cartesian point sets.

2.3 Formalizing the model

In order to formalize the attempts of the forgoing subchapter to model an incidence geometry for extended geographic objects for GIS, we introduce a number of predicates that characterize the Cartesian point sets P, Q and S qualitatively:

- *P is disc-shaped,* dSh(*P*)
- *S is line-shaped,* lSh(*S*)
- *P and Q are separated (not overlapping),* sep(*P*,*Q*)
- *P is incident with S,* inc(*P*,*S*)
- *S and S’ are equal,* eq(*S*,*S’*)

These predicates are qualitative linguistic descriptions and do not have a numerically precise meaning. As such, they are useful to formulate the conditions we want to impose on an interpretation of the connection operation in a GIS context:

1. If the Cartesian point sets *P* and *Q* are separated, their connection *S* should be incident with both, *P* and *Q*, and *S* should be line-shaped.
2. *S* should be unique. I.e., if *S’* is incident with *P* and *Q* and if *S’* is line-shaped, then *S* and *S’* should be equal.

This is a reformulation of the first and second incidence axiom as given in chapter 2.1. In this reformulation *l* is instantiated with the connection operator *S* and the terms *distinct*, *line*, *incident* and *equality* (uniqueness) are replaced by sep(), lSh(), inc() and eq(). For short, the above conditions can be written as follows:

1. $$sep(P,Q) \Rightarrow inc(S,P) \ \wedge \ inc(S,Q) \ \wedge lSh(S) \qquad (2)$$
2. $$\begin{aligned}&[inc(S,P) \ \wedge \ inc(S,Q) \ \wedge lSh(S)] \\ &\wedge [inc(S',P) \ \wedge \ inc(S',Q) \ \wedge lSh(S')] \Rightarrow eq(S,S')\end{aligned} \qquad (3)$$

In two-valued logic, predicates can assume either the truth value true (“1”), or the truth value false (“0”). For example, for the formula $inc(S,P) \ \wedge \ inc(S,Q) \ \wedge lSh(S)$ in (2) to be true, all three predicates must be true simultaneously.
The investigations of chapter 2.2 suggest that there is no canonical way to assign the truth value 1 simultaneously to all predicates and thereby make (2) and (3) satisfiable. In the next section, graded truth values are introduced to circumvent the problem of satisfiability.

3. FUZZY LOGIC

Fuzzy logic is a form of multi-valued logic, where the set of truth values comprises the interval [0,1] rather than the discrete set {0,1}. Fuzzy logic is derived from fuzzy set theory, which was introduced 1965 in the seminal paper (Zadeh, 1961) by Lotfi Zadeh. In a broader sense, fuzzy logic can be used as a tool for approximate reasoning (Zadeh, 1975).

3.1 Graded truth

In the context of basic fuzzy predicate logic, the predicates dSh(), lSh(),sep(), inc() and eq() defined in section 2.3 can be interpreted as fuzzy predicates: each of the (unary or binary) fuzzy predicates is represented by a (unary or binary) fuzzy relation that assigns to every member of the respective domain a truth value from [0,1]. For example, the unary predicate lSh() assigns every Cartesian point set *A* a value $\lambda \in [0,1]$ that expresses the degree to which *A* is line-shaped.

In the following, we will introduce possible numerical interpretations for each of the predicates. As preliminary terms we define for a set A of Cartesian points the minimal diameter

$$\phi_{\min}(A) = \min_t \left| ch(A) \cap \left\{ cg(ch(A)) + t \cdot R_\alpha \cdot (0,1)^T \mid t \in \mathbb{R} \right\} \right| \quad (5)$$

and the maximal diameter

$$\phi_{\max}(A) = \max_t \left| ch(A) \cap \left\{ cg(ch(A)) + t \cdot R_\alpha \cdot (0,1)^T \mid t \in \mathbb{R} \right\} \right| \quad (6)$$

where *ch(A)* and *cg(A)* denote the convex hull and the centre of gravity of *A*, respectively, and R_α denotes the rotation by angle α (see Figure 5a).

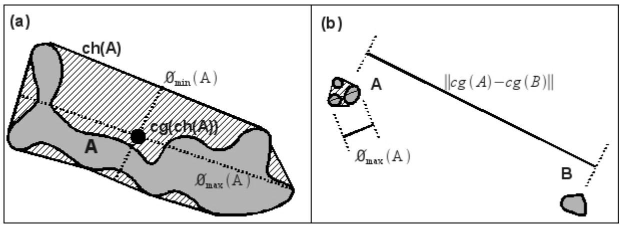


Figure 5. (a) Minimal and maximal diameter of a set *A* of Cartesian points. (b) Grade of separation sep(*A*,*B*) of *A* and *B*.

For a set *A* of Cartesian points let

$$dSh(A) = \frac{\phi_{\min}(A)}{\phi_{\max}(A)}$$
$$lSh(A) = 1 - dSh(A)$$
$$sep(A,B) = \max\left(0, 1 - \frac{\max\left(\phi_{\max}(A), \phi_{\max}(B)\right)}{\left\| cg(A) - cg(B) \right\|}\right) \quad (7)$$
$$inc(A,B) = \max\left(\frac{|A \cap B|}{|A|}, \frac{|A \cap B|}{|B|}\right)$$
$$eq(A,B) = \min\left(\frac{|A \cap B|}{|A|}, \frac{|A \cap B|}{|B|}\right).$$

Note that every logical and geometrical notion used in the original formulation of the incidence axioms corresponds to a graduated fuzzy predicate.
In order to avoid imposing additional information on the geometry that is not represented by the data, we choose to use the convex hull $P \bigcup Q = ch(P \cup Q)$ of the union of two extended points P and Q as the interpretation of the connection operation $\bigcup$.

Example: The opera house *O* and the Louvre museum *L* in Figure 1 have a grade of separation of sep(*O*,*L*)=0.8, whereas the objects A and C in Figure 2 have a grade of separation of sep(*A*,*B*)=0. The degree of line-shapedness amounts to $lSh(O \bigcup L) = 0.9$ and $lSh(A \bigcup C) = 0.3$, respectively.

3.2 Approximate Reasoning

Within the framework of fuzzy logic in the broader sense, propositions of two-valued logic, as introduced in section 2.3, can be restated in the form of fuzzy proposition.
For example the proposition " 'The extended points P and Q are separated' is true" – more concisely written as "sep(*P*,*Q*)=1" – was one of the starting points in this paper. Put into a fuzzy form, it can be reformulated as "The truth degree of 'The extended points P and Q are separated' is *high*", were *high* is a fuzzy membership function defined on the domain [0,1] (Figure 6). In short, "sep(*P*,*Q*) is *high*". A membership function "*high*" for a binary predicate is a fuzzy relation and can be constructed in a pertinently.

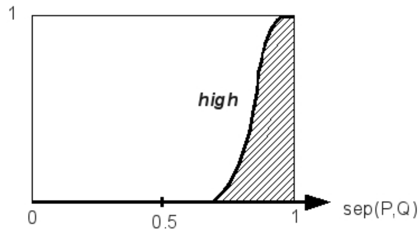


Figure 6. A fuzzy membership function for "*high*".

The conditions 1 and 2 of section 2.3 on the connection operation can now be formulated in the form of fuzzy rules:

1. IF sep(*P*,*Q*) is *high* THEN
   inc($P \bigcup Q$,*P*) is *high* AND
   inc($P \bigcup Q$,*Q*) is *high* AND
   lSh($P \bigcup Q$) is *high.*
2. IF inc(*S'*,*P*) is *high* AND
   inc(*S'*,*Q*) is *high* AND
   lSh(*S'*) is *high* THEN
   eq($P \bigcup Q$,*S'*) is *high.*

Fuzzy propositions allow for a certain tolerance in the degree of validity of predicates and thereby leave room for a trade-off of properties. At the same time linguistic expressions such as "*high*" impose a fuzzy constraint on the data via their membership function: they are flexible constraints.
Given a set of extended geographic objects, the fuzzy rules 1 and 2 can be used for an approximate pre-evaluation of potential operations and their behaviour. The fuzzy model carries the grade of validity of every intermediary reasoning step and can give warning whenever – after a number of reasoning steps - the grade of validity falls below a certain threshold.
The use of linguistic expressions is especially useful in the context of tourist information systems and decision support systems in general: With appropriate choices of linguistic terms, the interpretability of system messages for lay users is dramatically increased.

4. CONCLUSIONS

4.1 Conclusions

We have shown that straight forward interpretations of the connection of extended points do not satisfy the incidence axioms of Euclidean geometry in a strict sense. Yet, the geometric behaviour of extended objects can be approximately described by a fuzzified version of the incidence axioms. To describe the approximate behaviour, the framework of approximate reasoning is used. The proposed mechanism provides the possibility to warn users, when geometric operations are performed that are not significant due to error accumulation.
The use of fuzzy reasoning trades accuracy against speed, simplicity and interpretability by lay users. In the context of

tourist in formation systems, these characteristics are clearly advantageous.

4.2 Discussion and further work

The membership function given in Figure 6 is a somewhat arbitrary choice. A parameter optimization for finding the membership function that best corresponds to the real behaviour of the model is subject of future work. Furthermore, we will investigate the sensitivity of the fuzzy model with respect to the choice of different interpretations of geometric primitives.

In the definitions (5),(6) and (7) of the fuzzy predicates dSh(), lSh(),sep(), inc() and eq(), every Cartesian point set that is interpreted as an extended point is wrapped with its convex hull before further processing. The wrapping provides a smoothing of the geometric properties of the point sets and thereby ensures that the problem is not numerically ill-posed, i.e. small changes in the input do not result in huge jumps in the output. Following this rudimentary continuity argument, it can be expected that the fuzzy rules given in section 3.2 provide an acceptable approximation of the real behaviour of the model.

The set of incidence axioms discussed in the present article is only one out of fours primitive relations of Hilbert's axiomatic system of Euclidean geometry. In future work, we will extend the set of interpretations of relations between extended points and lines to the following relations: betweenness (order), parallelism and congruence.

After a model validation, the fuzzy rules corresponding to the axioms can be used to derive approximate geometric theorems in a way pertinent to deriving crisp theorems.

4.3 Acknowledgement

The research was funded by the Austrian Marshall Plan Foundation.

5. REFERENCES

Bennett, B., Cohn, A., Torrini, P., Hazarika, S.M., 2000. A Foundation for Region-Based Qualitative Geometry. *Proceedings of ECAI-2000*, pp. 204--208.

Dilo, A., 2006. Representation of and reasoning with vagueness in spatial information - A system for handling vague objects. Doctoral dissertation. C.T. de Wit Graduate School for Production Ecology and Resource Conservation (PE&RC) in Wageningen University, the Netherlands.

Dutta, S., 1990. Qualitative Spatial Reasoning: A Semi-quantitative Approach Using Fuzzy Logic. *Lecture notes in computer science* 409, pp. 345-364.

Frank, A., 1992. Qualitative spatial reasoning about distances and directions in geographic space.

Freksa, C., 1991. Qualitative spatial reasoning. In: D.M. Mark & A.U. Frank (eds.), *Cognitive and Linguistic Aspects of Geographic Space*, pp. 361-372.

Gerla, G., 1990. Pointless metric spaces. *The Journal of Symbolic Logic*, 55(1), pp. 207-219.

Hilbert, D., 1962. *Grundlagen der Geometrie*. Teubner Studienbuecher Mathematik, 1962.

Renz, J., Nebel, B., 2007. Qualitative spatial reasoning using constraint calculi. *Handbook of Spatial Logics*. Springer Netherlands, pp. 161-215.

Schmidt, H.J., 1979. Axiomatic characterization of physical geometry. *Lecture Notes in Physics*, Springer, Berlin.

Schmidtke, H.R., 2005. Eine axiomatische Charakterisierung räumlicher Granularität: formale Grundlagen detailgrad-abhängiger Objekt- und Raumrepräsentation. Doctoral dissertation, Universität Hamburg, Fachbereich Informatik, 2005.

Tarski, A., 1956. *Logics, Semantics, Mathematics*. Oxford University press, Oxford, pp. 24-30.

Zadeh, L.A., 1965. Fuzzy sets. *Fuzzy Systems and AI Reports and Letters*, 61, pp. 129—136.

Zadeh, L.A., 1975. The concept of a linguistic variable and its application to approximate reasoning I. *Information Science*, 8, pp. 199-250.